# Validating Aircraft Models in the Gap Metric

Andrei Dorobantu, Gary Balas, and Tryphon Georgiou

Aerospace Engineering & Mechanics
Electrical and Computer Engineering
University of Minnesota

Gloverfest
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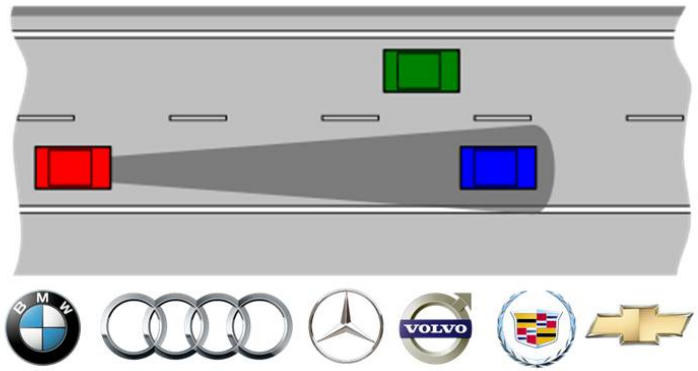
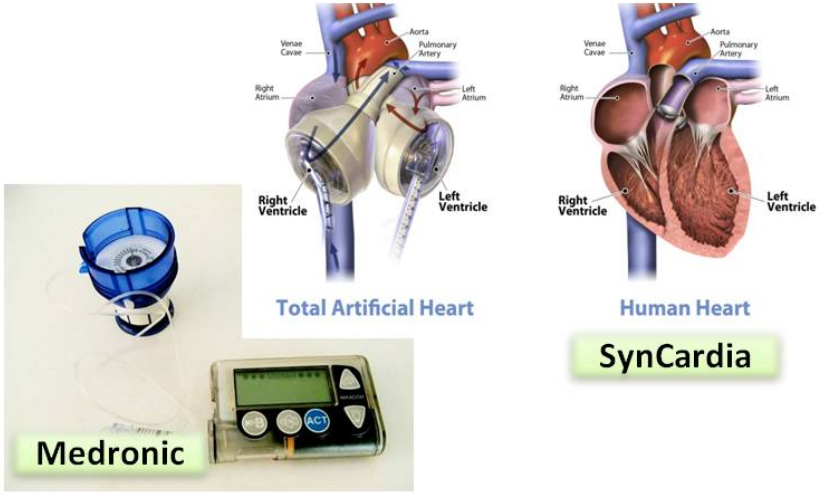
# Validating Aircraft Models: Mind the Gap

# Low-Cost, Safety-Critical Systems




(a) Intelligent vehicles

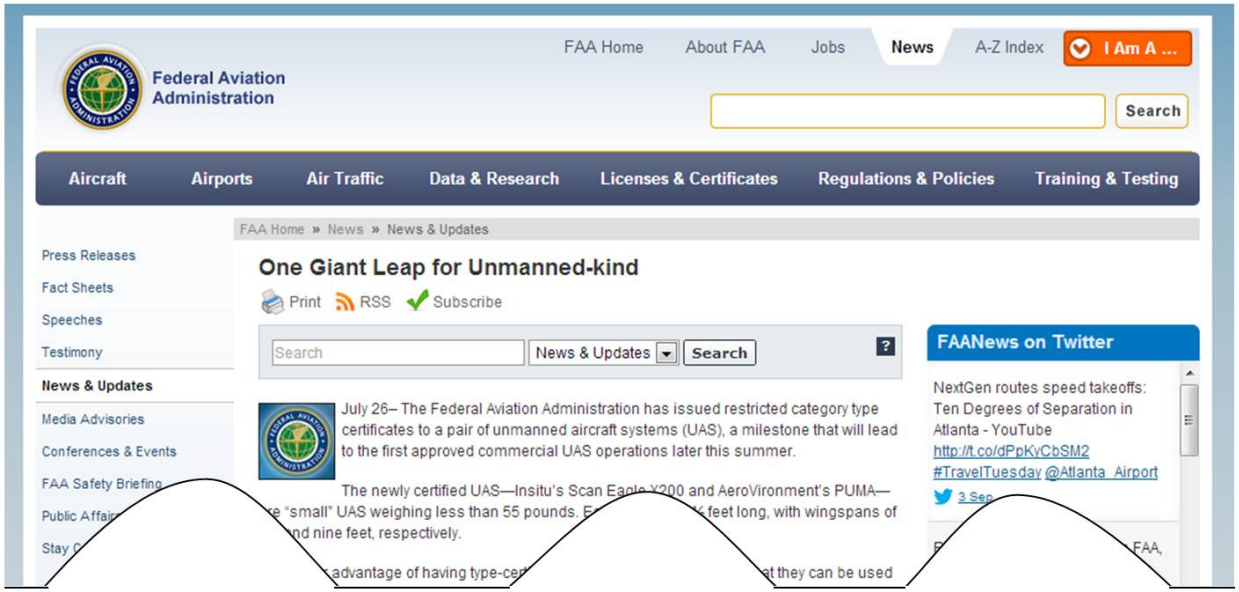



(b) Medical devices


(c) Unmanned aircraft systems (UAS)

Sources: Wikipedia, AeroVironment, Insitu

# Low-Cost, Safety-Critical Systems



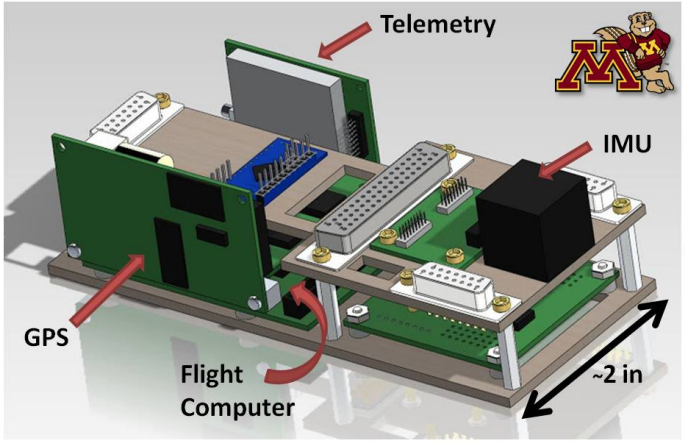


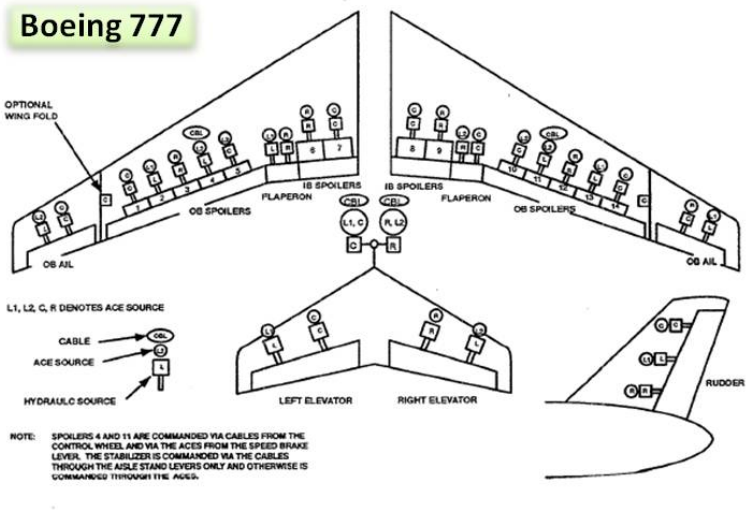
(c) Unmanned aircraft systems (UAS)

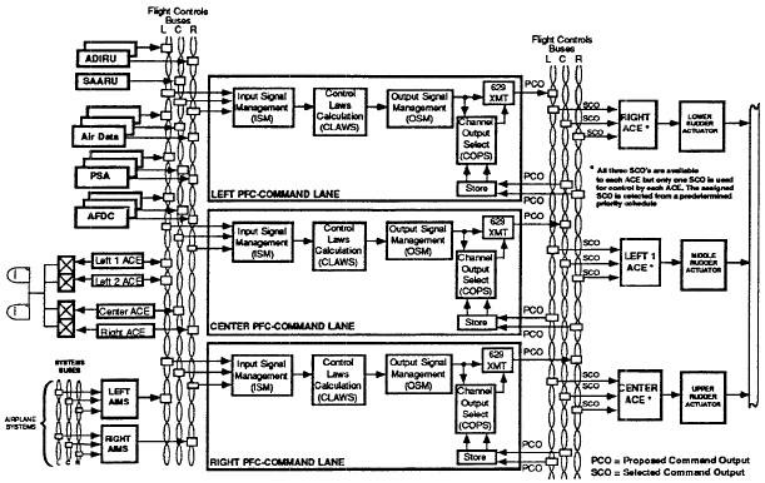
Sources: Wikipedia, AeroVironment, Insitu

## UAS vs. traditional aircraft

1. Limited budget
2. Restricted physical space
3. Restricted weight
4. Short development time




Source: Y. C. Yeh, *Triple-triple redundant 777 primary flight computer*, 1996

# Efficient and Affordable Models

Significant cost savings can be achieved if Model Based Design is done properly; however, accurate models are required.

- "There are known knowns. These are things we know that we know. There are known unknowns. That is to say, there are things that we know we don't know. But there are also unknown unknowns. There are things we don't know we don't know." Donald Rumsfeld

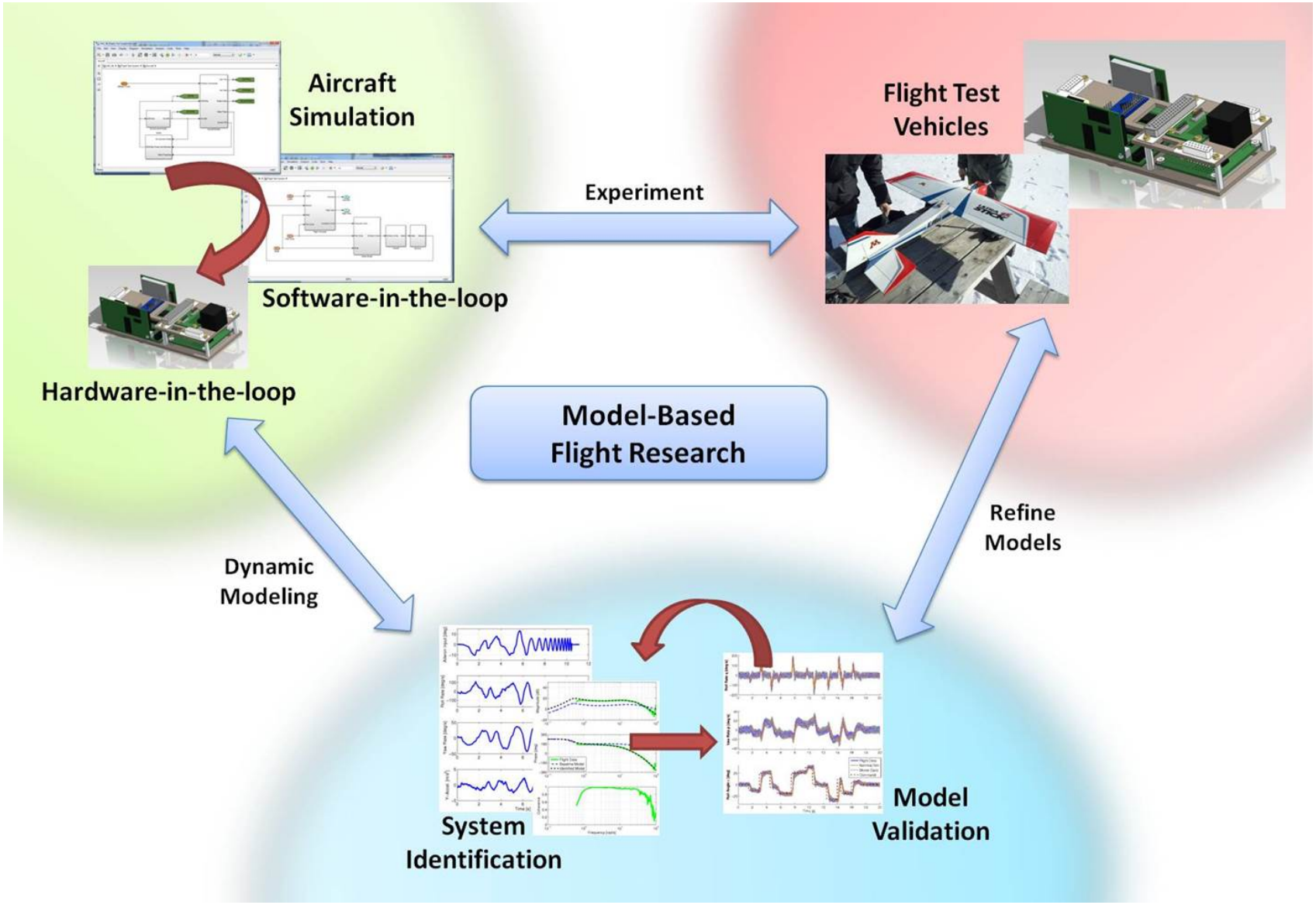
Consider the F-35 aircraft, designed extensively based on mathematical models, which is facing challenges:

- "Test pilots say some of the F-35's delays can be traced to the concept that computer simulation and modeling would recognize many of the F-35 design problems right away, instead of the old-fashioned method of testing: in the air [1]."

Cost effective flight testing provides opportunities to verify assumptions, refine models.

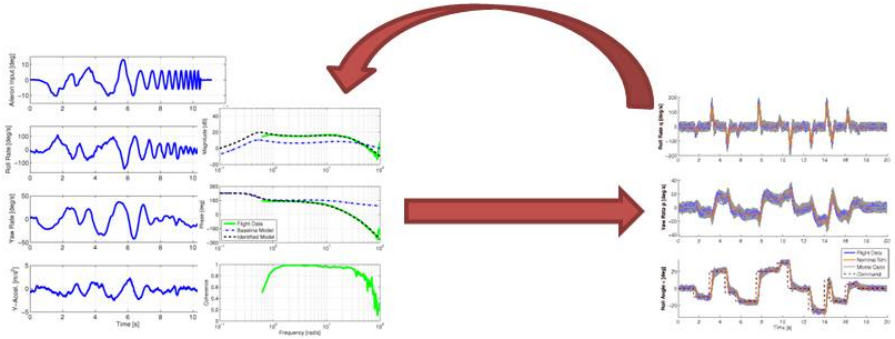
1. W.J. Hennigan and Ralph Vartabedian, "F-35 fighter jet struggles to take off," L.A. Times, 6/12/2013.

# Low-Cost UAS Test Platform



# System Identification & Model Validation

**System Identification**



**Model Validation**

1. Established techniques
   - Time domain
   - Frequency domain
   - Software available
2. Opportunity:
   - Engineering insight for low-cost systems

1. Emerging techniques
   - LMI-based optimization
   - Statistics/residuals
   - Ad-hoc application
2. **Opportunity:**
   - Rigorous tools tied to robustness requirements

## Main Idea

- How do aerospace engineers validate models using flight data?
  - Bring mathematical rigor to the process
  - **Ultimate goal:**
    - Reduce development cost
    - Fast, rigorous tools to validate models and control systems

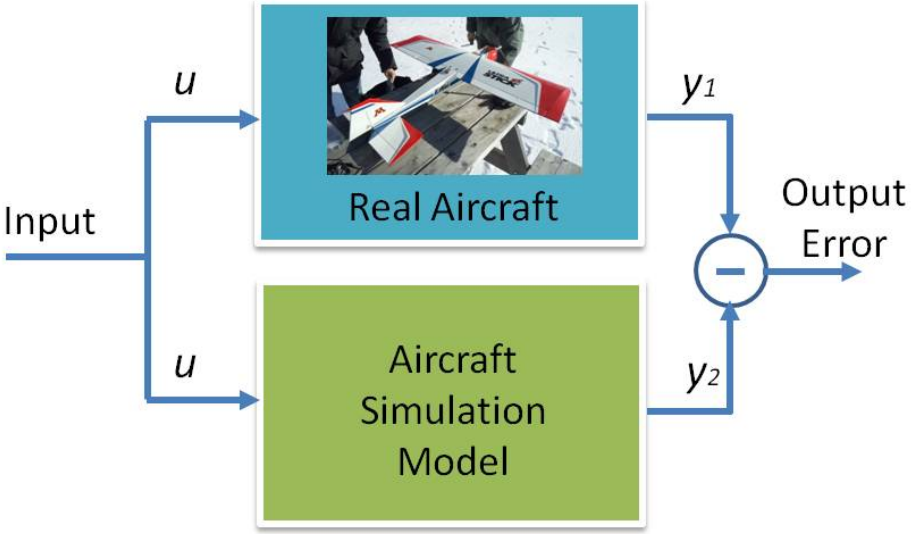
- **Theil's Inequality Coefficient**
  - Originally proposed as a tool for economic forecasting
  - Adopted for system analysis by the aerospace community

$$\mathrm{TIC}(y_1, y_2) = \frac{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_{1_i} - y_{2_i})^2}}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_{1_i})^2} + \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_{2_i})^2}}$$

Henri Theil

Source: Intl. Statistics Institute

**Aerospace engineering perspective:**

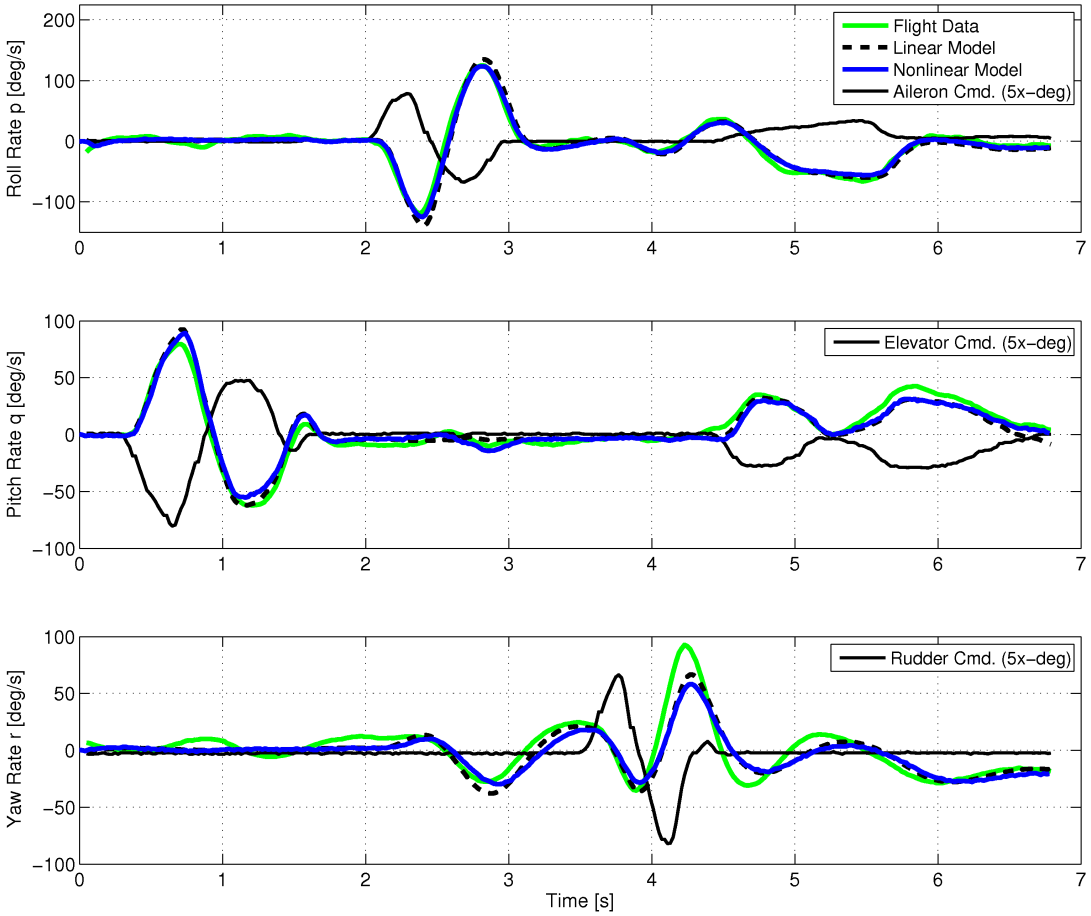



1. Ranges from 0 to 1
2. TIC $= 0 \rightarrow$ perfect model
3. TIC $= 1 \rightarrow$ no correlation
4. Lower TIC $\rightarrow$ better model

$$\text{TIC}(y_1, y_2) = \frac{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_{1_i} - y_{2_i})^2}}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_{1_i})^2} + \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_{2_i})^2}}$$

# Theil's Inequality Coefficient

$$\mathrm{TIC}(y_1, y_2) = \frac{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_{1_i} - y_{2_i})^2}}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_{1_i})^2} + \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_{2_i})^2}}$$




- $\mathrm{TIC}_{roll} = 0.07$
- $\mathrm{TIC}_{pitch} = 0.12$
- $\mathrm{TIC}_{yaw} = 0.26$

# ONE IDEA

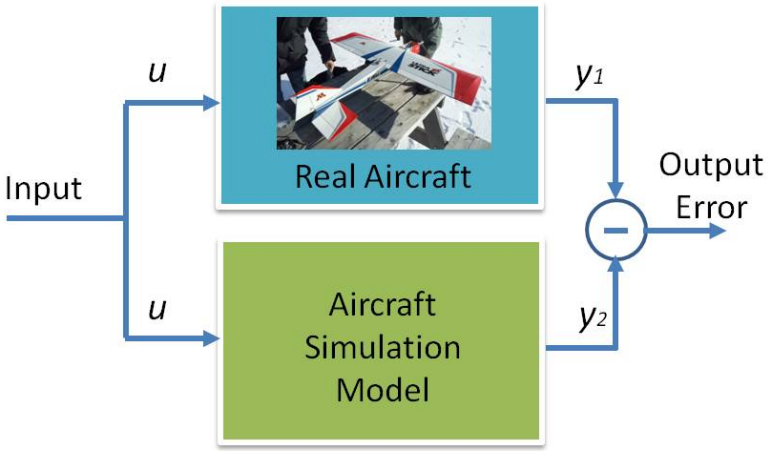
**Theil's Inequality Coefficient $\approx$ Gap Metric**

**Modified TIC:**

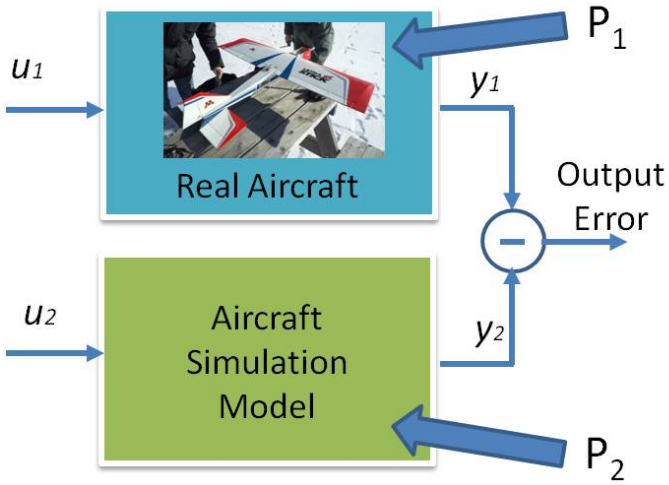
$$\mathrm{TIC}_{sys}(y_1, y_2, u) = \frac{\left\| \begin{bmatrix} u \\ y_1 \end{bmatrix} - \begin{bmatrix} u \\ y_2 \end{bmatrix} \right\|_2}{\left\| \begin{bmatrix} u \\ y_1 \end{bmatrix} \right\|_2 + \left\| \begin{bmatrix} u \\ y_2 \end{bmatrix} \right\|_2}$$

**Gap Metric:**

$$\vec{\delta}(P_1, P_2) = \sup_{||u_1||_2 \leq 1} \inf_{u_2} \frac{\left\| \begin{bmatrix} u_1 \\ y_1 \end{bmatrix} - \begin{bmatrix} u_2 \\ y_2 \end{bmatrix} \right\|_2}{\left\| \begin{bmatrix} u_1 \\ y_1 \end{bmatrix} \right\|_2}$$



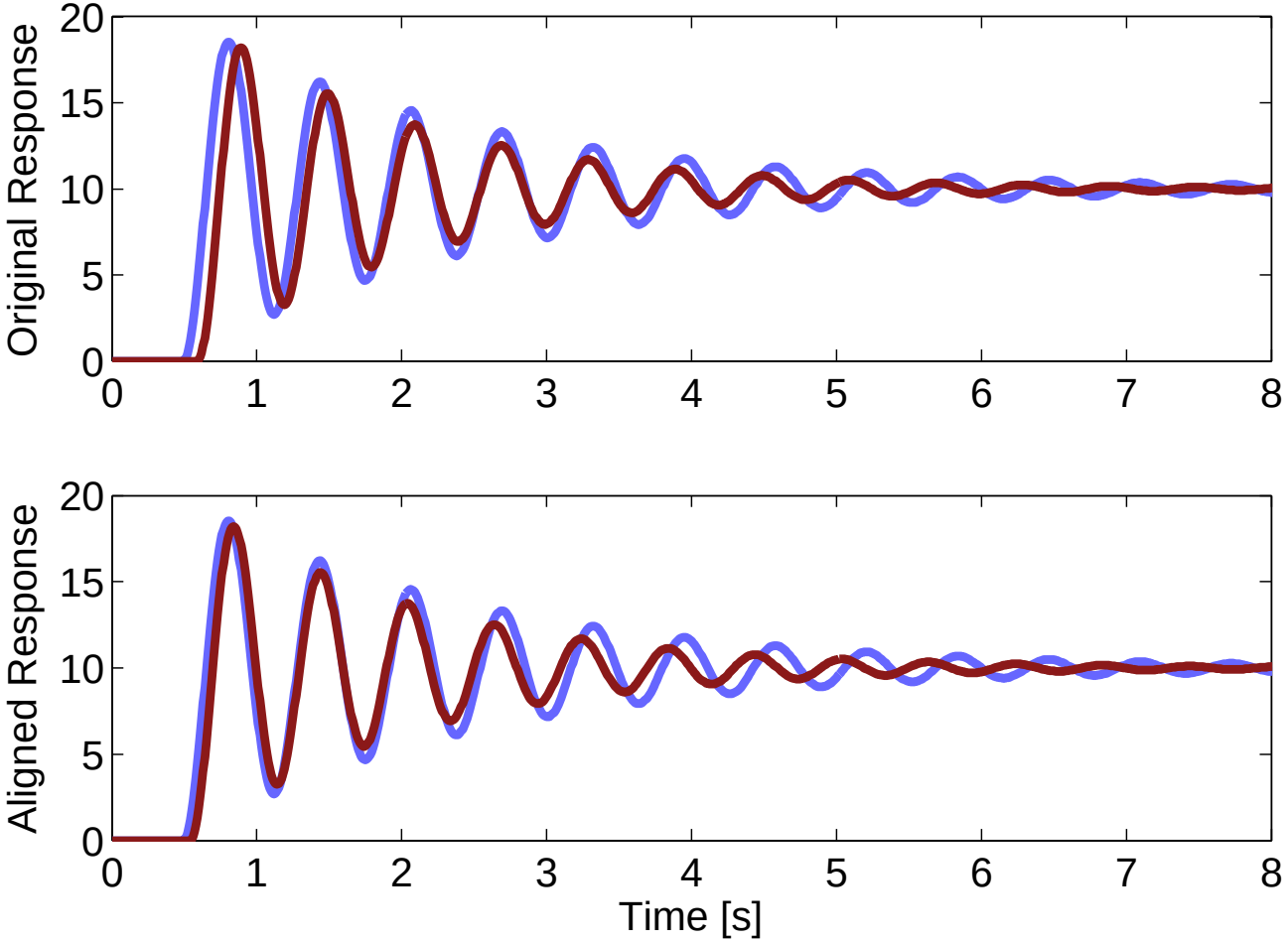

(a) TIC diagram




(b) Gap diagram

# The Gap Metric: Aligning Input/Outputs

**Aerospace engineers already do this:**




- TIC $= 0.077$
- TIC $= 0.047$

# The Gap Metric: An Evolution

## ❶ Time domain gap metric:

$$\vec{\delta}(P_1, P_2) = \sup_{||u_1||_2 \leq 1} \inf_{u_2} \frac{\left\| \begin{bmatrix} u_1 \\ y_1 \end{bmatrix} - \begin{bmatrix} u_2 \\ y_2 \end{bmatrix} \right\|_2}{\left\| \begin{bmatrix} u_1 \\ y_1 \end{bmatrix} \right\|_2}$$

G. Zames & A. El-Sakkary

## ❷ Frequency domain gap metric:

$$\vec{\delta}(P_1, P_2) = \inf_{Q \in \mathcal{H}_\infty} \| G_1 - G_2 Q \|_\infty$$

- For linear systems $P_1$ and $P_2$
- $G_1$ and $G_2$ are graph operators, represent input–output signals

T. Georgiou & M. Smith

## ❸ $\nu$-gap metric:

$$\delta_\nu(P_1, P_2) = \left\| (1 + P_2 P_2^*)^{-\frac{1}{2}} (P_2 - P_1) (1 + P_1 P_1^*)^{-\frac{1}{2}} \right\|_\infty$$

- Applicable to frequency responses

G. Vinnicombe (Source: IEEE)

# The Gap Metric and Robustness

1. **Gap metric born in the robust control framework:**
   - Guaranteed stability margins for closed-loop systems
   - Coprime factor uncertainty
   - $P_2(s) \in P_\Delta(s) \stackrel{\text{def}}{=} P_1(s)\,(1+\delta_1)/(1-\delta_2)$
   - $|\delta_i| \leq \epsilon \leq 1$

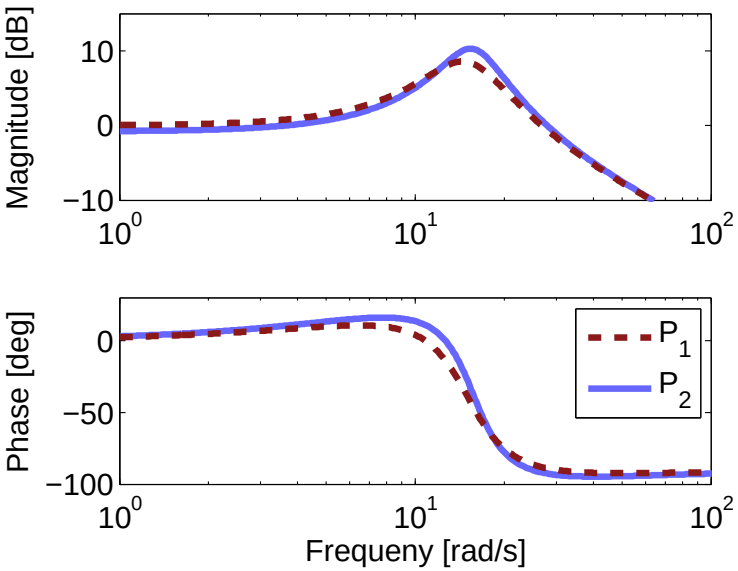
2. **Related to classical robustness margins:**
   - Gain margin $= 20 \log_{10} \frac{1+\epsilon}{1-\epsilon}$
   - Phase margin $= 2 \arcsin \epsilon$
   - Disk margin $= \frac{2\,\epsilon}{1-\epsilon^2}$

K. Glover, G. Vinnicombe, and G. Papageorgiou
Source: IEEE

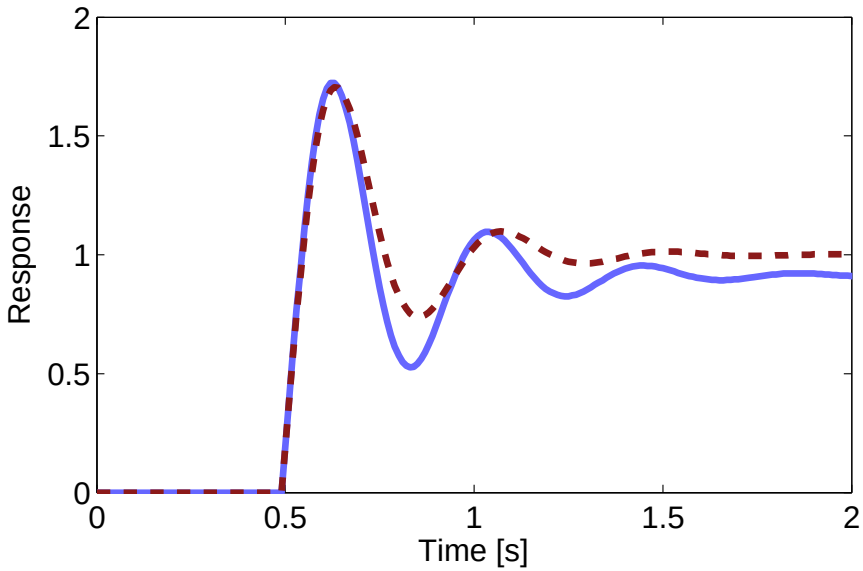
**Model validation of simple second-order systems:**

- Quantify model similarity using the gap metric

$$P_1(s) = \frac{18.75s + 225}{s^2 + 9s + 225} \qquad P_2(s) = \frac{18.75s + 225}{s^2 + 7.22s + 246.5}$$



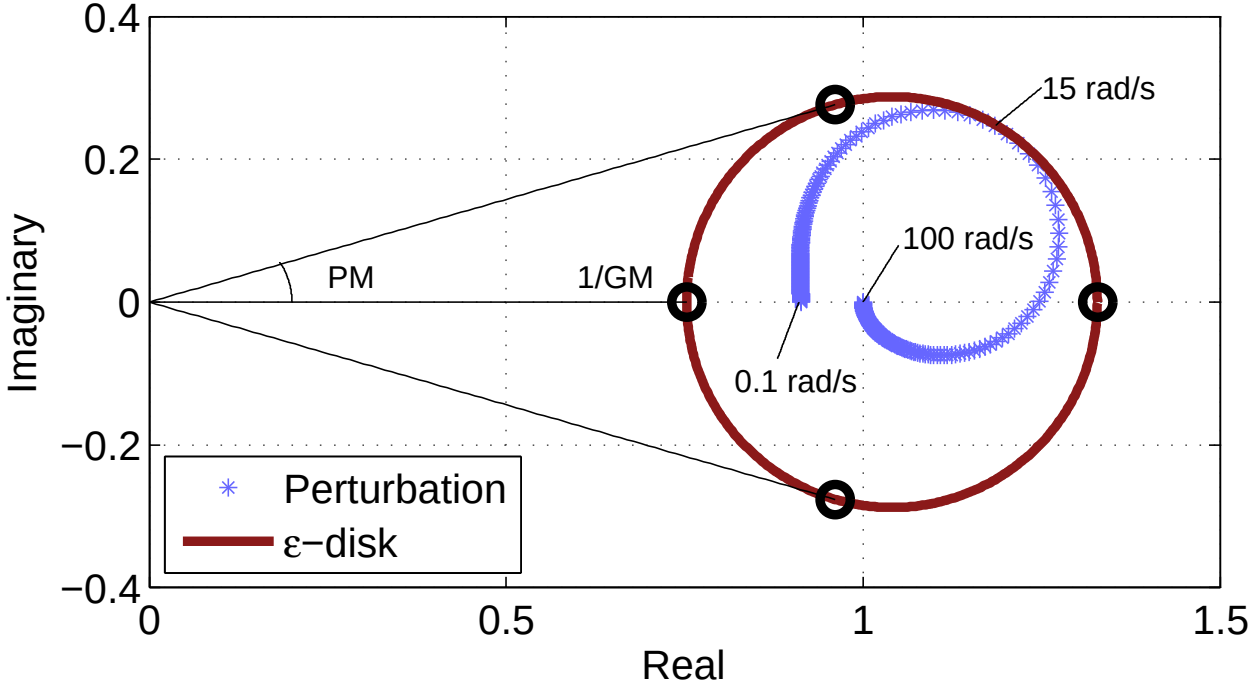

(a) Frequency responses




(b) Step responses

# Simple Example

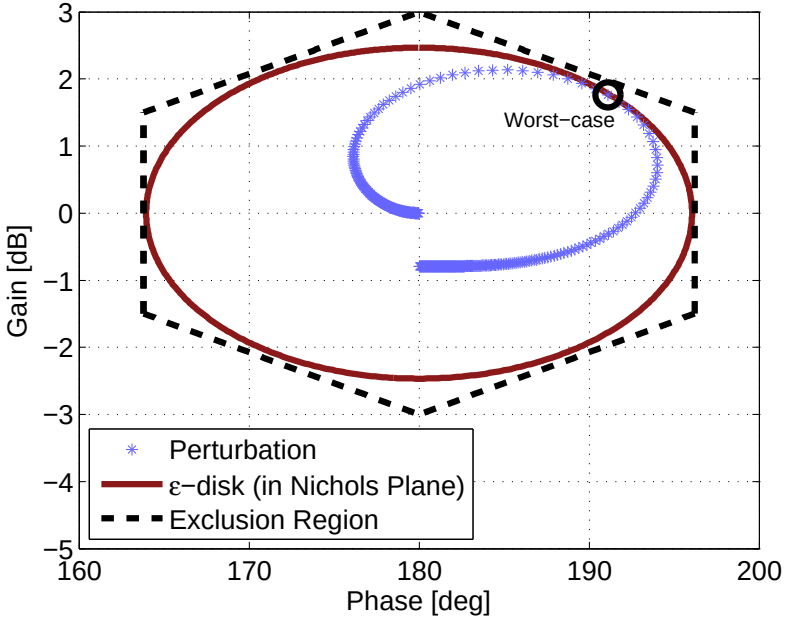
$$P_1(s) = \frac{18.75s + 225}{s^2 + 9s + 225} \qquad P_2(s) = \frac{18.75s + 225}{s^2 + 7.22s + 246.5}$$



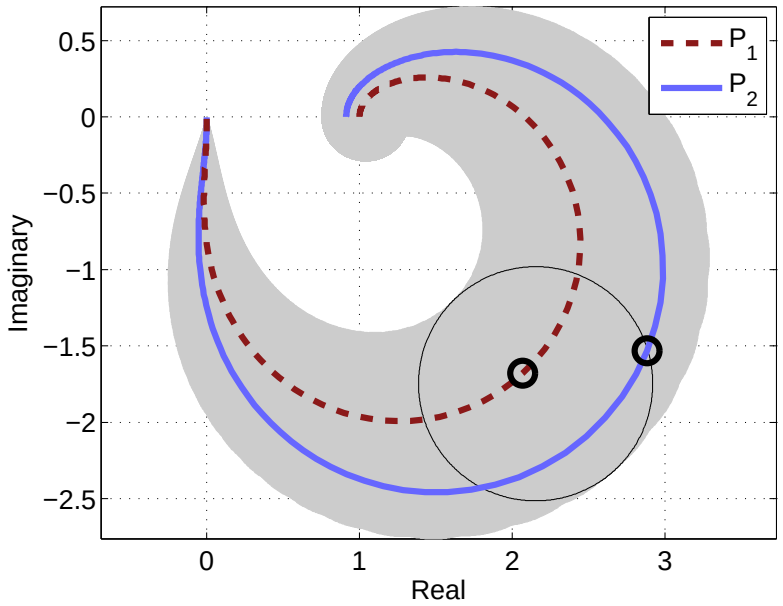

Gap Metric $= 0.14$ Simultaneous Gain/Phase Margin $= 2.48$ dB, 16.2 deg

# Simple Example

$$P_1(s) = \frac{18.75s + 225}{s^2 + 9s + 225} \qquad P_2(s) = \frac{18.75s + 225}{s^2 + 7.22s + 246.5}$$




(a) Nichols Diagram

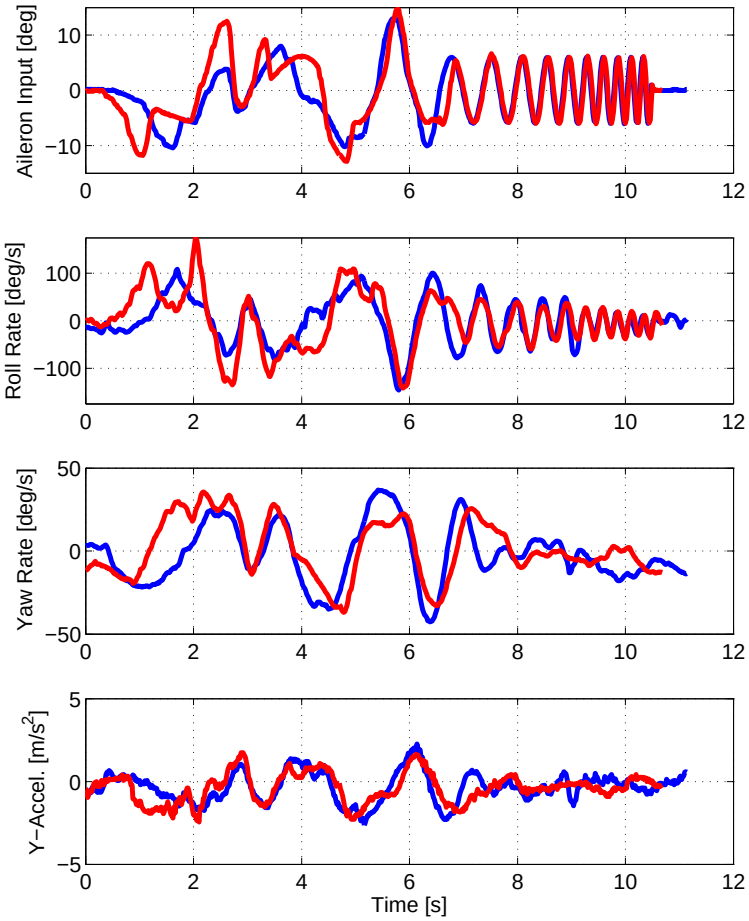



(b) Nyquist Diagram

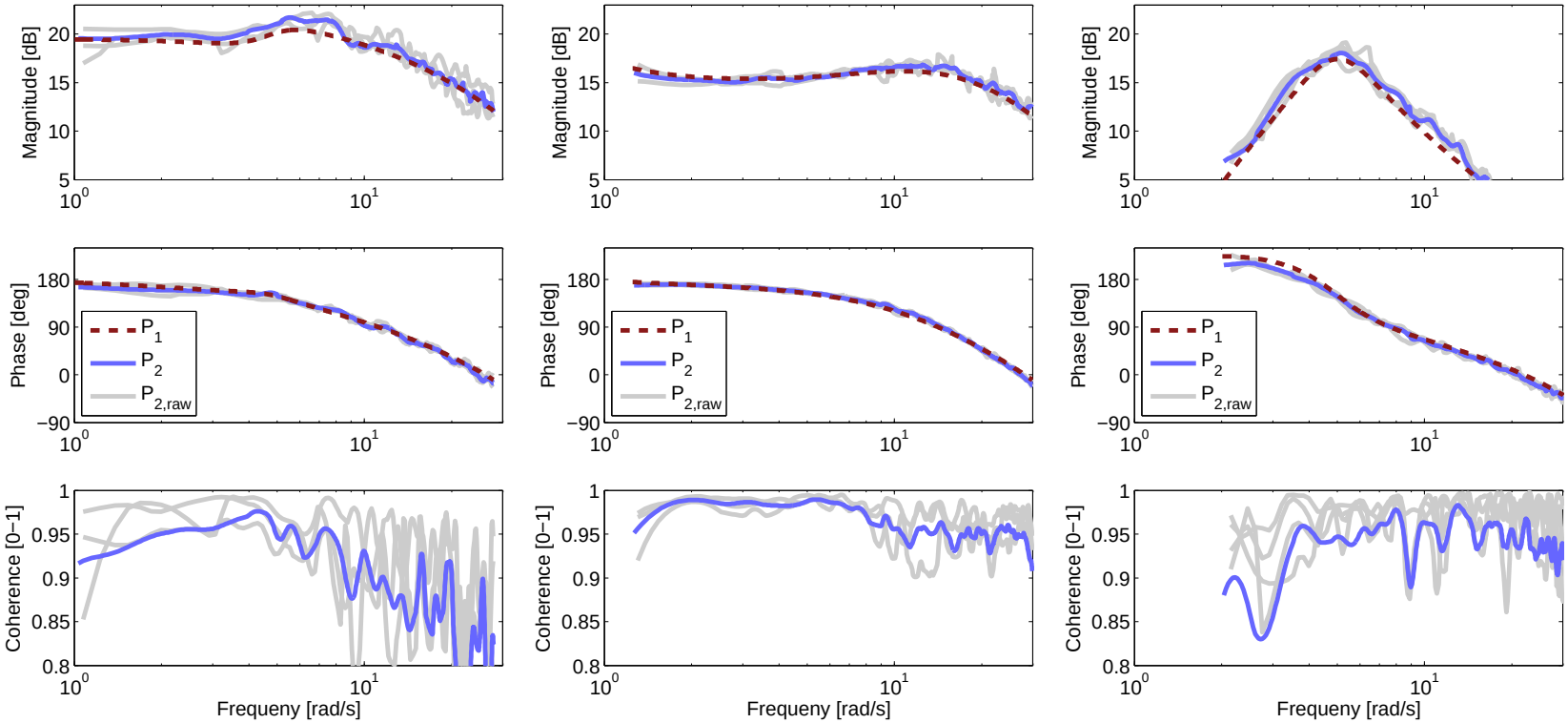
Gap Metric $= 0.14$ Simultaneous Gain/Phase Margin $= 2.48$ dB, 16.2 deg

## Real flight data and models:

- 6 flight tests
- Chirp-style inputs

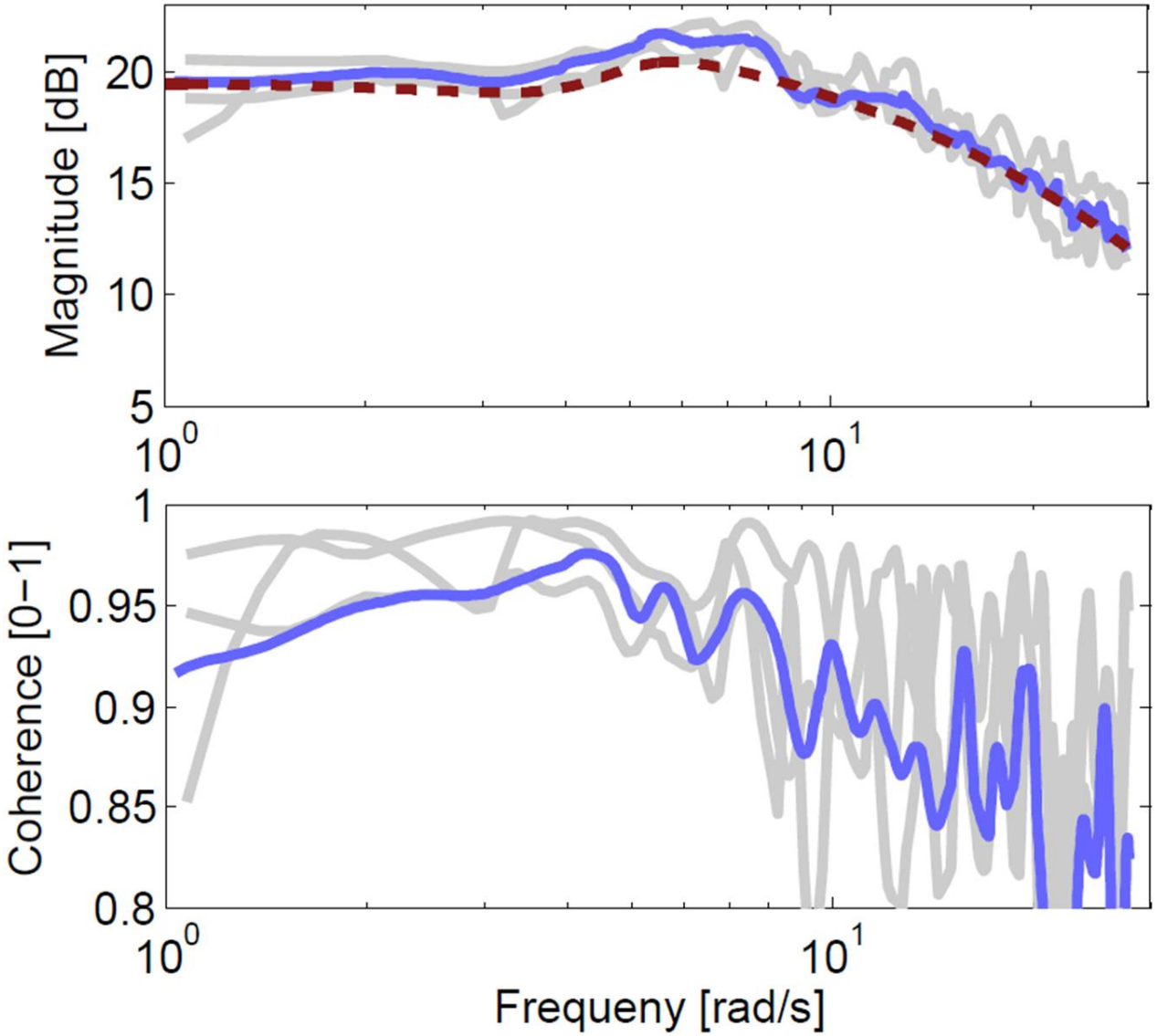


# UAS Application

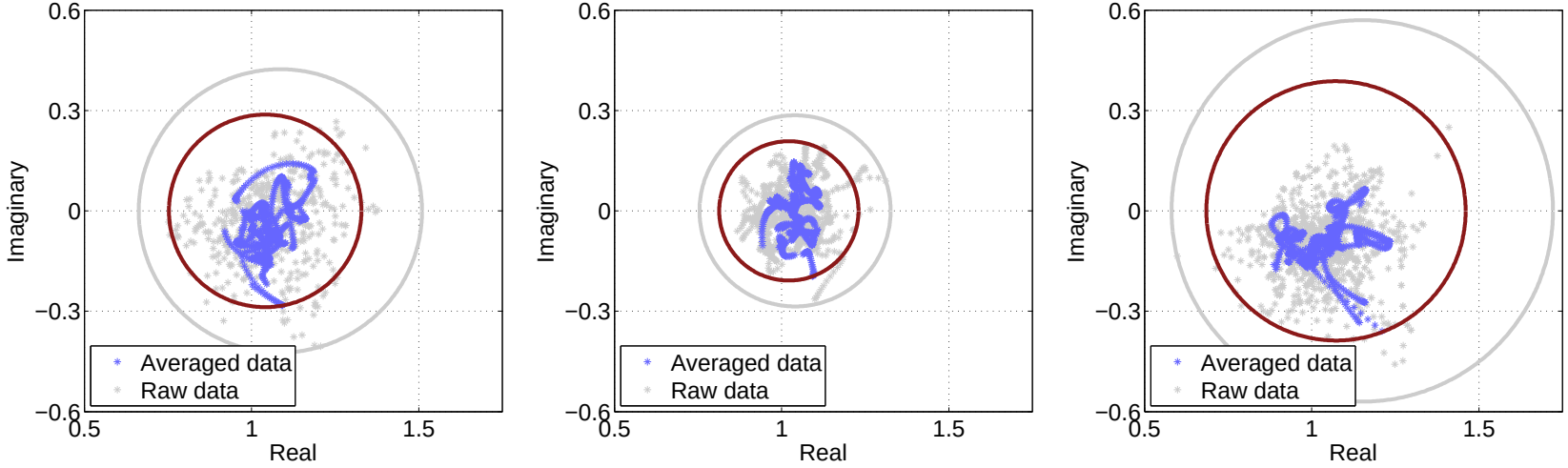



(a) Aileron to roll rate (b) Elevator to pitch rate (c) Rudder to yaw rate

# UAS Application: $\delta_{ail} \rightarrow p$



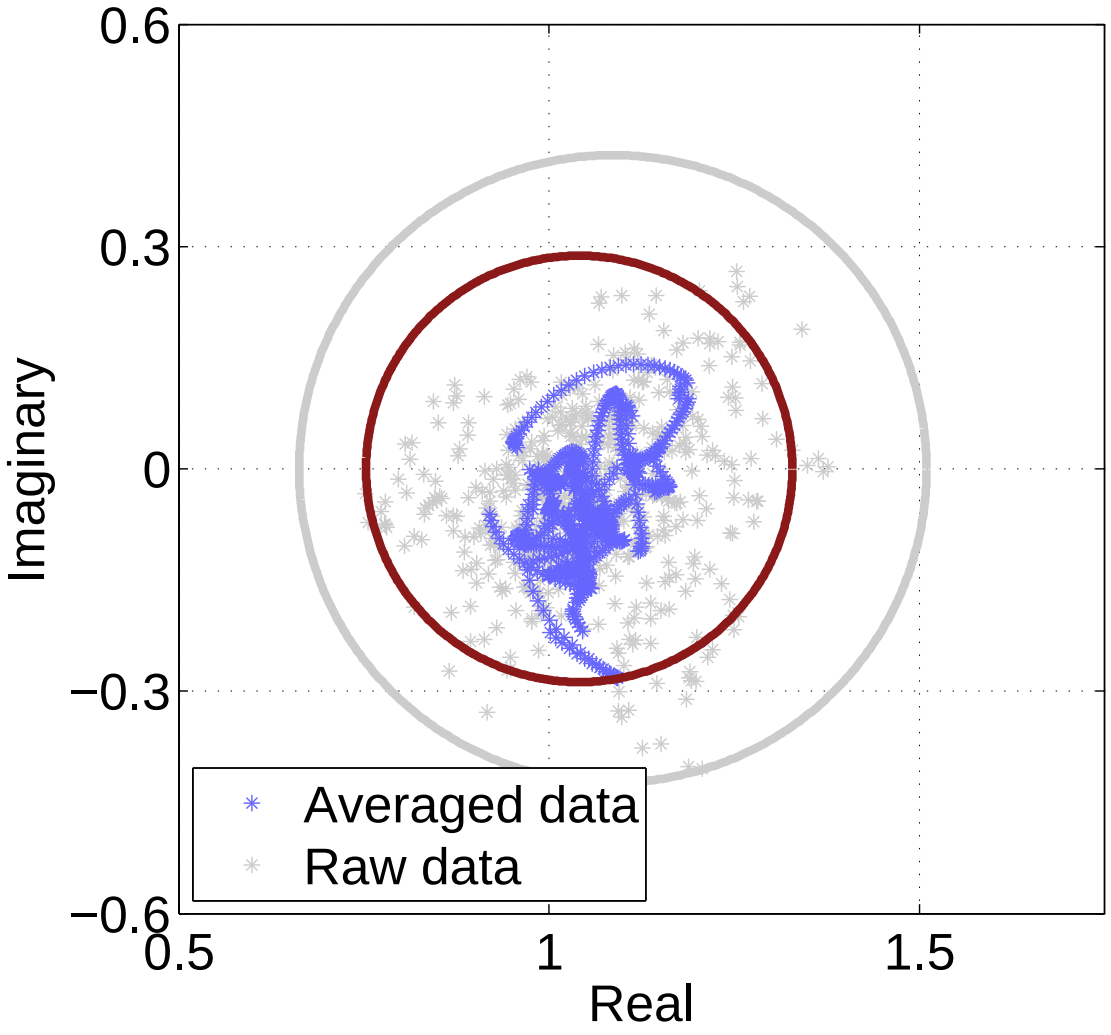
# UAS Application: Model Validation



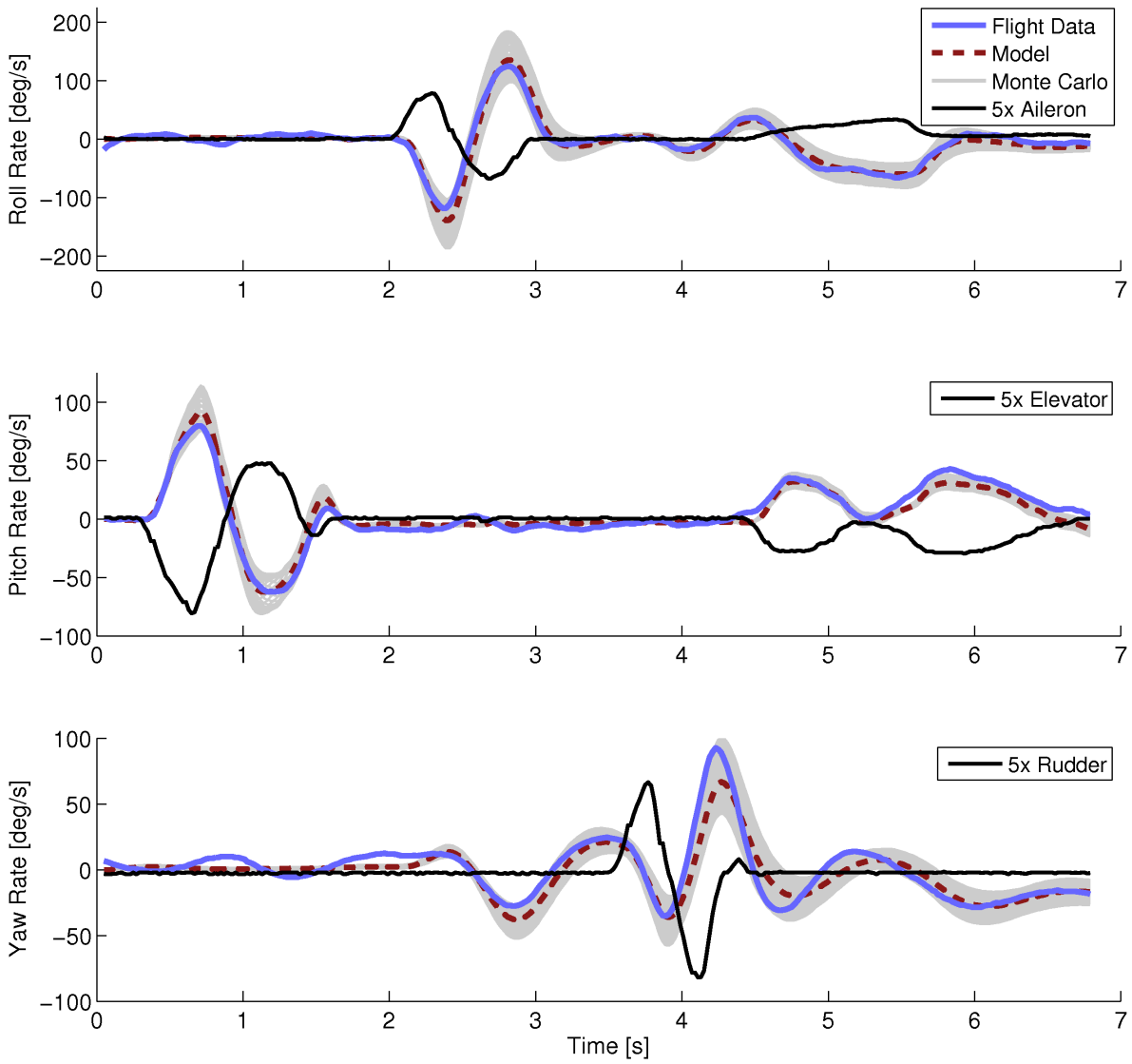

(a) Aileron to roll rate (b) Elevator to pitch rate (c) Rudder to yaw rate

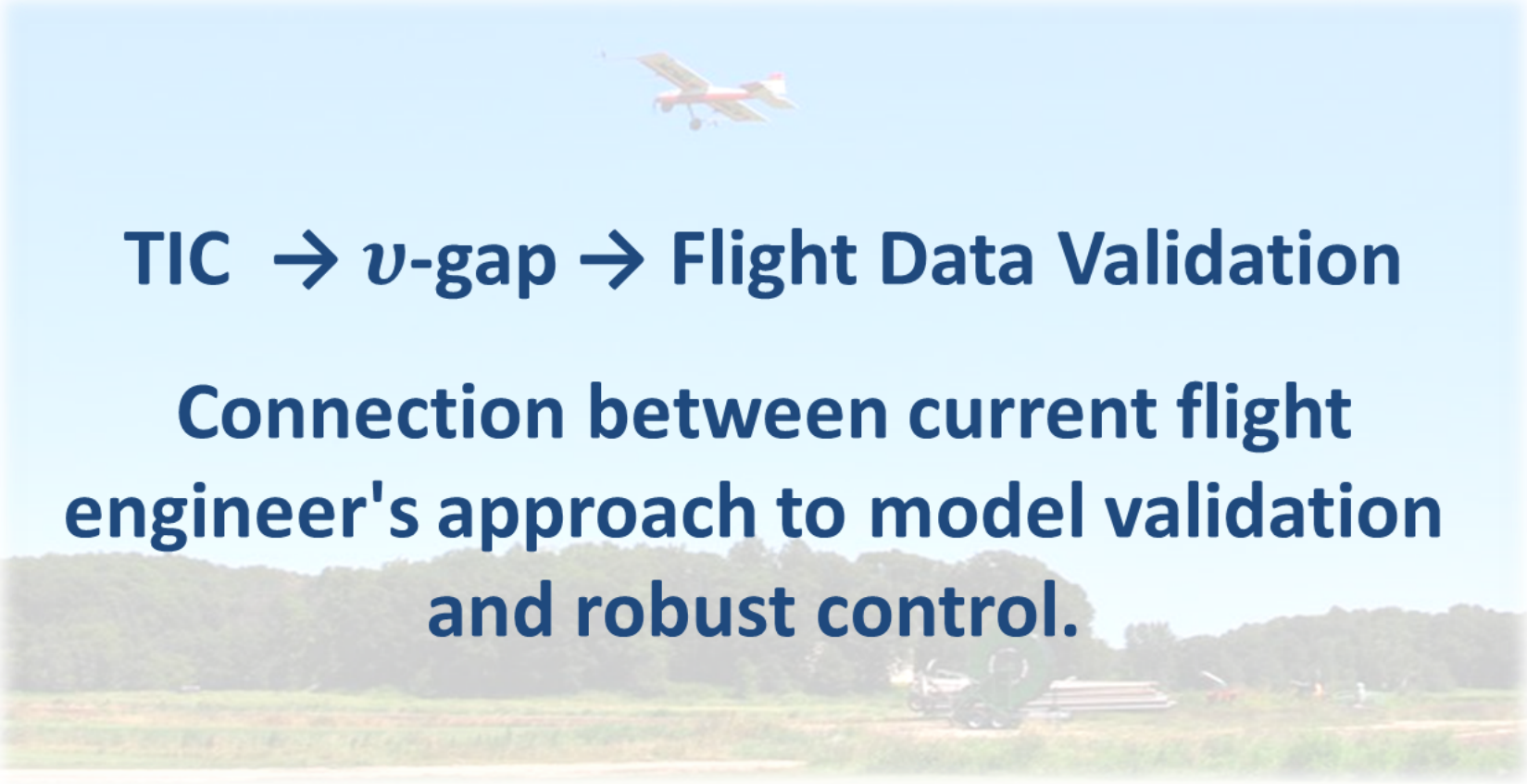
| | | Roll Rate | | Pitch Rate | | Yaw Rate | |
|---|---|---|---|---|---|---|---|
| | | Raw | Avg. | Raw | Avg. | Raw | Avg. |
| Model Quality | Gap Metric $\epsilon$ | 0.20 | 0.14 | 0.14 | 0.10 | 0.27 | 0.19 |
| Controller Requirements | Gain Mar. [dB] | 3.58 | 2.47 | 2.45 | 1.80 | 4.70 | 3.29 |
| | Phase Mar. [deg] | 23.43 | 16.21 | 16.10 | 11.82 | 30.73 | 21.29 |
| | Disk Mar. | 0.42 | 0.28 | 0.29 | 0.21 | 0.57 | 0.39 |

# UAS Application: Model Validation (Zoom)



# UAS Application: Time Domain Verification





TIC → $\upsilon$-gap → Flight Data Validation

Connection between current flight engineer's approach to model validation and robust control.