# Fast Model Predictive Control (MPC)

Manfred Morari

with thanks to
Colin Jones, Paul Goulart,
Alex Domahidi, Stefan Richter
and many other collaborators

Automatic Control Laboratory, ETH Zürich

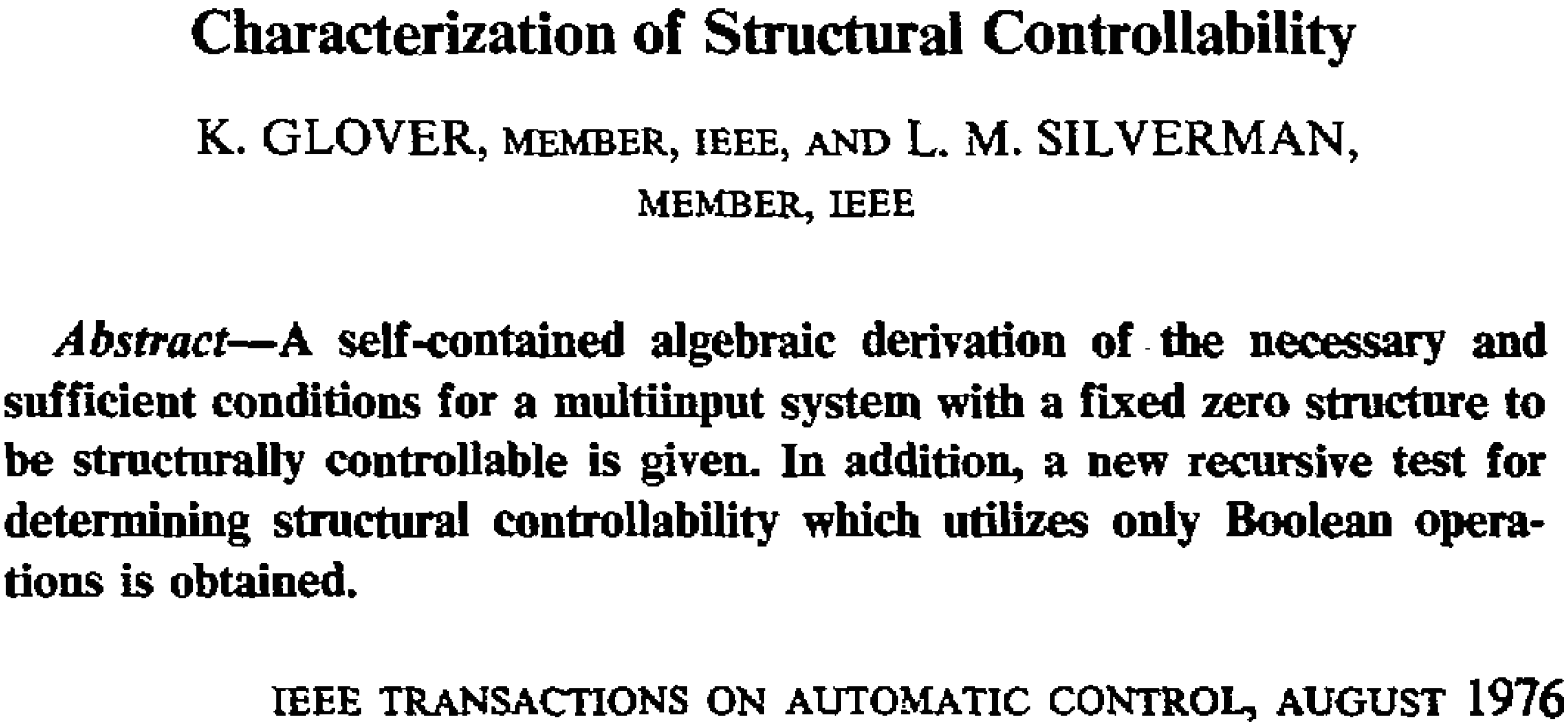
# Fast Model Predictive Control (MPC)

Manfred Morari

with thanks to
Colin Jones, Paul Goulart,
Alex Domahidi, Stefan Richter
and many other collaborators

Automatic Control Laboratory, ETH Zürich

# My first “encounter” with Keith….


**Characterization of Structural Controllability**

K. GLOVER, MEMBER, IEEE, AND L. M. SILVERMAN, MEMBER, IEEE

*Abstract*—A self-contained algebraic derivation of the necessary and sufficient conditions for a multiinput system with a fixed zero structure to be structurally controllable is given. In addition, a new recursive test for determining structural controllability which utilizes only Boolean operations is obtained.

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, AUGUST 1976

Published: 20.09.13

Campus

# Lino Guzzella appointed President of ETH Zurich

**The Swiss Federal Council has appointed Lino Guzzella, ETH Rector and Professor of Thermotronics, as the future ETH President. He will be taking over from Ralph Eichler who is retiring at the end of 2014. By taking this decision, the Swiss Government has endorsed the unanimous proposal made by the ETH Board.**

ETH President Ralph Eichler (r) congratulates his successor on his election. "I am very happy and proud at the confidence which the Federal Council and ETH Board have placed in me", Lino Guzzella comments. He will be taking up his duties on 1 January 2015.

# Fast Model Predictive Control (MPC)

Manfred Morari

with thanks to
Colin Jones, Paul Goulart,
Alex Domahidi, Stefan Richter
and many other collaborators

Automatic Control Laboratory, ETH Zürich

# Synthesis of Optimal Control Laws

### Infinite-Horizon Optimal Control

$$J^*(x) = \min_{u_i \in U} \sum_{i=0}^{\infty} l(x_i, u_i)$$
$$\text{s.t. } x_{i+1} = f(x_i, u_i)$$
$$x_i \in X$$

### Dynamic Programming

$$J^\star(x) = \min_u \; l(x,u) + J^\star(f(x,u))$$
$$\text{s.t. } (f(x,u),u) \in X \times U$$

- Challenge is computation!

# Synthesis of Optimal Control Laws

## Infinite-Horizon Optimal Control

$$J^*(x) = \min_{u_i \in U} \sum_{i=0}^{\infty} l(x_i, u_i)$$
$$\text{s.t. } x_{i+1} = f(x_i, u_i)$$
$$x_i \in X$$

## Dynamic Programming

$$J^\star(x) = \min_u \; l(x,u) + J^\star(f(x,u))$$
$$\text{s.t. } (f(x,u), u) \in X \times U$$

- Challenge is computation!
- Closed-form solution for linear systems, no constraints only: LQR,…

# Synthesis of Optimal Control Laws

**Infinite-Horizon Optimal Control**

$$J^*(x) = \min_{u_i \in U} \sum_{i=0}^{\infty} l(x_i, u_i)$$
$$\text{s.t. } x_{i+1} = f(x_i, u_i)$$
$$x_i \in X$$

**Dynamic Programming**

$$J^\star(x) = \min_u \; l(x,u) + J^\star(f(x,u))$$
$$\text{s.t. } (f(x,u),u) \in X \times U$$

Explicit calculation of control law $u^\star(x)$ *offline*

**Model Predictive Control**

$$J^*(x_0) = \min_{u_i} \sum_{i=0}^{N} l(x_i, u_i) + V_f(x_N)$$
$$\text{s.t. } (x_i, u_i) \in X \times U, \;\; x_N \in X_f$$
$$x_{i+1} = f(x_i, u_i)$$

*Online* optimization problem defines control action $u_0^\star(x)$

# Model Predictive Control : Properties

Theory is well-established

Mayne, Rawlings, Rao, Scokaert (2000), *Automatica*

“MPC: Stability & Optimality (Survey Paper). “

- **Recursive feasibility**: Input and state constraints are satisfied
- **Stability** of the closed-loop system
  - $J^*(x)$ is a convex Lyapunov function

- **Assuming** the real-time optimization problem is solved to ε-optimality

# Embedded Model Predictive Control

## Traditional MPC

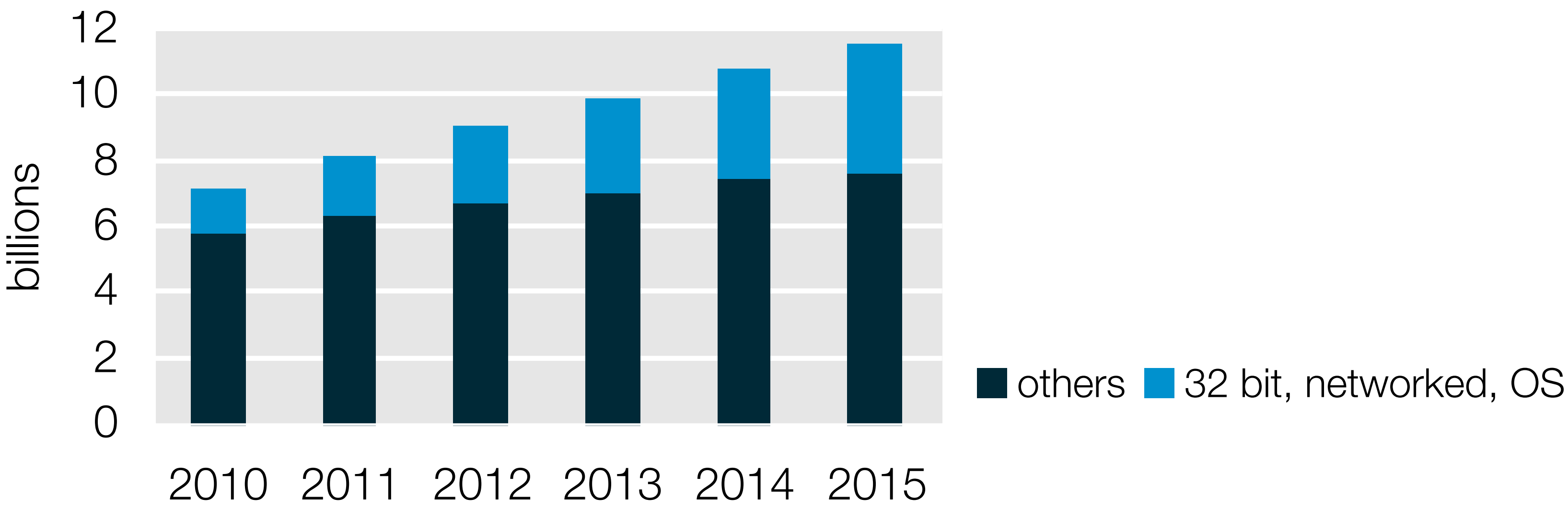
- Successful in process industries
- Sampling times of minutes
- Powerful computing platforms

## Embedded MPC

- Small, high performance plants
- Sampling times of ms to ns
- Limited embedded platform

# Embedded Systems by the Numbers

Worldwide unit shipments of embedded computing platforms:




Source: *Intelligent Systems: The Next Big Opportunity*, IDC 2011

ARM's 32-bit embedded systems growing at 20% p.a.

Source: Keith Clarke, ARM Vice President, *Keynote at CDNLive*, May 2013

**New opportunity for automated decision making based on optimization**

# Verifiable Control Synthesis

| Offline | Online |
|---|---|
| Explicit MPC | $1^{st}$ Order–Fast Gradient |
| Approx. Explicit MPC | Interior Point Opt. |

# Verifiable Control Synthesis

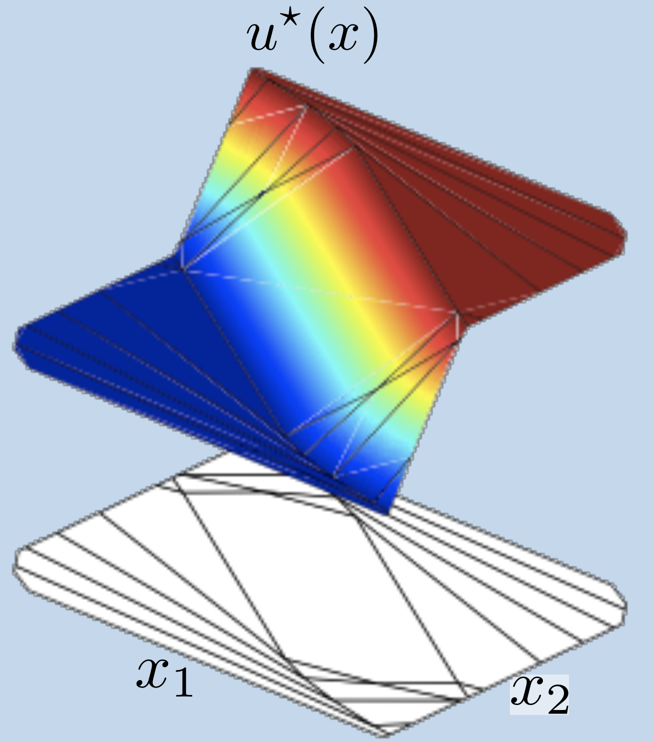
| Offline | Online |
|---|---|
| Explicit MPC | 1st Order–Fast Gradient |
| Approx. Explicit MPC | Interior Point Opt. |

# Explicit MPC : Online => Offline Processing

- Optimization problem is parameterized by state
- Control law piecewise affine for linear systems / constraints
- Pre-compute control law as function of state $x$ (parametric optimization)

Result : Online computation dramatically reduced

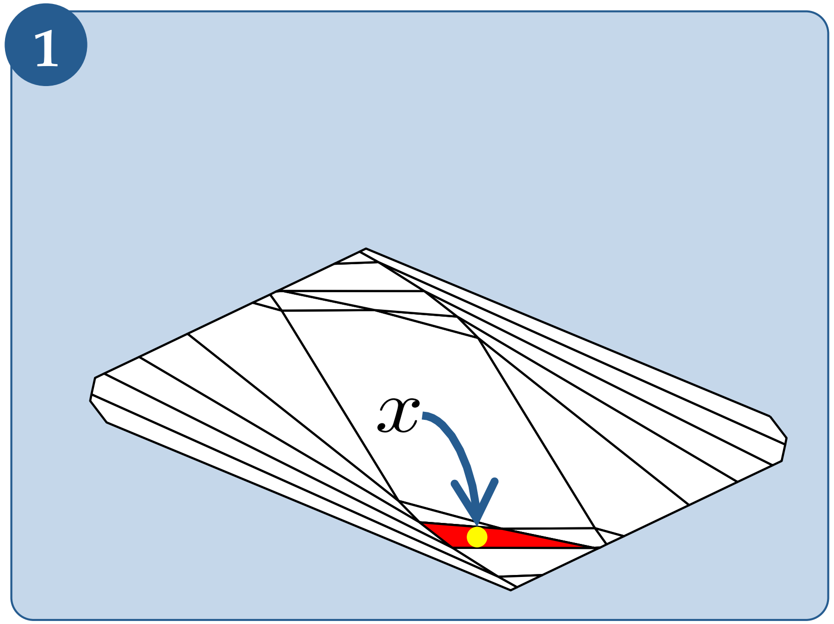
$$
\begin{aligned}
u^\star(x_0) = \operatorname*{argmin}_{u_i} \; & \sum_{i=0}^{N} l(x_i, u_i) + V_f(x_N) \\
\text{s.t. } & (x_i, u_i) \in X \times U \\
& x_{i+1} = f(x_i, u_i) \\
& x_N \in X_f
\end{aligned}
$$

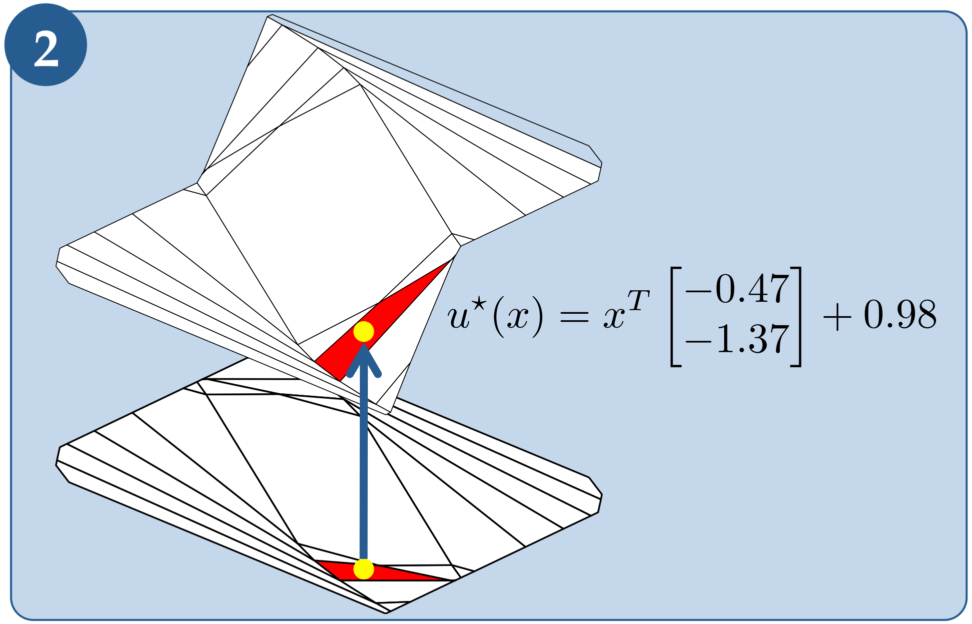



*[M.M. Seron, J.A. De Doná and G.C. Goodwin, 2000]*
*[T.A. Johansen, I. Peterson and O. Slupphaug, 2000]*
*[A. Bemporad, M. Morari, V. Dua and E.N. Pistokopoulos, 2000]*

# Explicit MPC : Fast online evaluation

- Online evaluation reduced to:
  1. Point location
  2. Evaluation of affine function
- Online complexity is governed by point location
  - Function of number of regions in cell complex
  - Milli- to microseconds possible if small number of regions



# Verifiable Control Synthesis

| Offline | Online |
| --- | --- |
| **Explicit MPC**<br>• Small problems<br>• Simple<br>• High speed | **$1^{st}$ Order–Fast Gradient** |
| **Approx. Explicit MPC** | **Interior Point Opt.** |

# Verifiable Control Synthesis

| Offline | Online |
|---|---|
| **Explicit MPC**<br>• Small problems<br>• Simple<br>• High speed | **1st Order–Fast Gradient** |
| **Approx. Explicit MPC**<br>• Medium size<br>• Specified complexity<br>• High speed | **Interior Point Opt.** |

# Verifiable Control Synthesis

| Offline | Online |
|---|---|
| **Explicit MPC**<br>• Small problems<br>• Simple<br>• High speed | **1st Order–Fast Gradient** |
| **Approx. Explicit MPC**<br>• Medium size<br>• Specified complexity<br>• High speed | **Interior Point Opt.** |

# Fast Gradient Method : Time bound to ε-optimality

- ε-solution in $i_{\min}$ steps

$$i_{\min} \geq \left\lceil \frac{\ln \frac{\delta}{\epsilon}}{-\ln\left(1 - \sqrt{\frac{1}{\kappa}}\right)} \right\rceil$$

- κ condition number
- δ measure of ***initial error***

*[Y. Nesterov, 1983]*
*[S. Richter, C.N. Jones and M. Morari, CDC 2009]*

| Cold start |
|---|
| $\delta = LR^2/2$ |
| • R : radius of feasible set<br>• Easy to compute |

| Warm start |
|---|
| $\delta = 2 \max_{x \in \mathbb{X}_0} J_N(U_w; x) - J_N^*(x)$ |
| • $U_w$ : Warm start solution<br>• Worst distance measured in terms of initial cost<br>• Hard to compute |

# Verifiable Control Synthesis

| Offline | Online |
| --- | --- |
| **Explicit MPC**<br>• Small problems<br>• Simple<br>• High speed | **$1^{st}$ Order–Fast Gradient**<br>• Large problems<br>• Less simple<br>• Deterministic computation time |
| **Approx. Explicit MPC**<br>• Medium size<br>• Specified complexity<br>• High speed | **Interior Point Opt.** |

# Verifiable Control Synthesis

| Offline | Online |
| --- | --- |
| **Explicit MPC**<br>• Small problems<br>• Simple<br>• High speed | **1st Order–Fast Gradient**<br>• Large problems<br>• Less simple<br>• Deterministic computation time |
| **Approx. Explicit MPC**<br>• Medium size<br>• Specified complexity<br>• High speed | **Interior Point Opt.** |

# Convex Multistage Problem

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{N} \frac{1}{2} v_i^T H_i v_i + f_i^T v_i \\
\text{subject to} \quad & \underline{v}_i \leq v_i \leq \bar{v}_i \\
& A_i v_i \leq b_i \\
& v_i^T Q_{i,j} v_i + l_{i,j}^T v_i \leq r_{i,j} \\
& C_i v_i + D_{i+1} v_{i+1} = c_i
\end{aligned}
$$

separable objective

upper/lower bounds

affine inequalities

quadratic inequalities

affine equalities, each coupling only two consecutive variables

where $H_i, Q_{i,j} \succeq 0$ and $A_i$ has full row rank

Captures MPC, MHE, portfolio optimization, spline optimization, etc.

# The FORCES Code Generator

## Multistage QCQP

$$
\begin{aligned}
\min \quad & \textstyle\sum_{i=1}^{N} \frac{1}{2} v_i^T H_i v_i + f_i^T v_i \\
\text{s.t.} \quad & \underline{z}_i \leq v_i \leq \bar{v}_i \\
& A_i v_i \leq b_i \\
& v_i^T Q_{i,j} v_i + l_{i,j}^T v_i \leq r_{i,j} \\
& C_i v_i + D_{i+1} v_{i+1} = c_i
\end{aligned}
$$

?

## Embedded Hardware

## Problem description

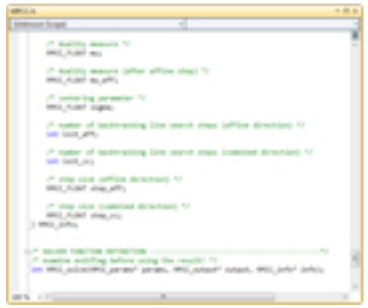
```
stage = MultiStageProblem(N+1);
for i = 1:N+1
% dimensions
stages(i).dims.n = 10;
stages(i).dims.r = 5;
stages(i).dims.lb = 3;

% cost
stages(i).cost.H = Hi;
stages(i).cost.f = fi;

% inequalities
stages(i).ineq.b.lbidx = 3:5;
stages(i).ineq.b.lb = zeros(3,1);

% equalities
stages(i).eq.C = Ci;
stages(i).eq.c = ci;
stages(i).eq.D = Di;
end
generateCode(stages);
```

FORCES

- C code generation of primal-dual Mehrotra interior point solvers
- LPs, QPs, QCQPs
- Parametric problems
- Multi-core platforms
- Library-free
- Available: forces.ethz.ch

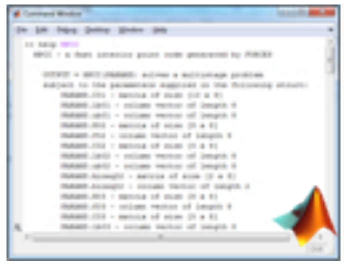
## Generated Code

Solver (ANSI-C)



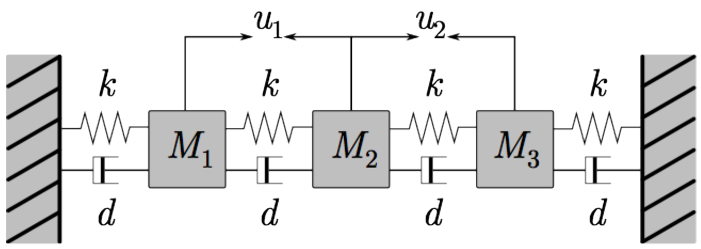
solver.h
solver.c
solver.m
solvermex.c
makemex

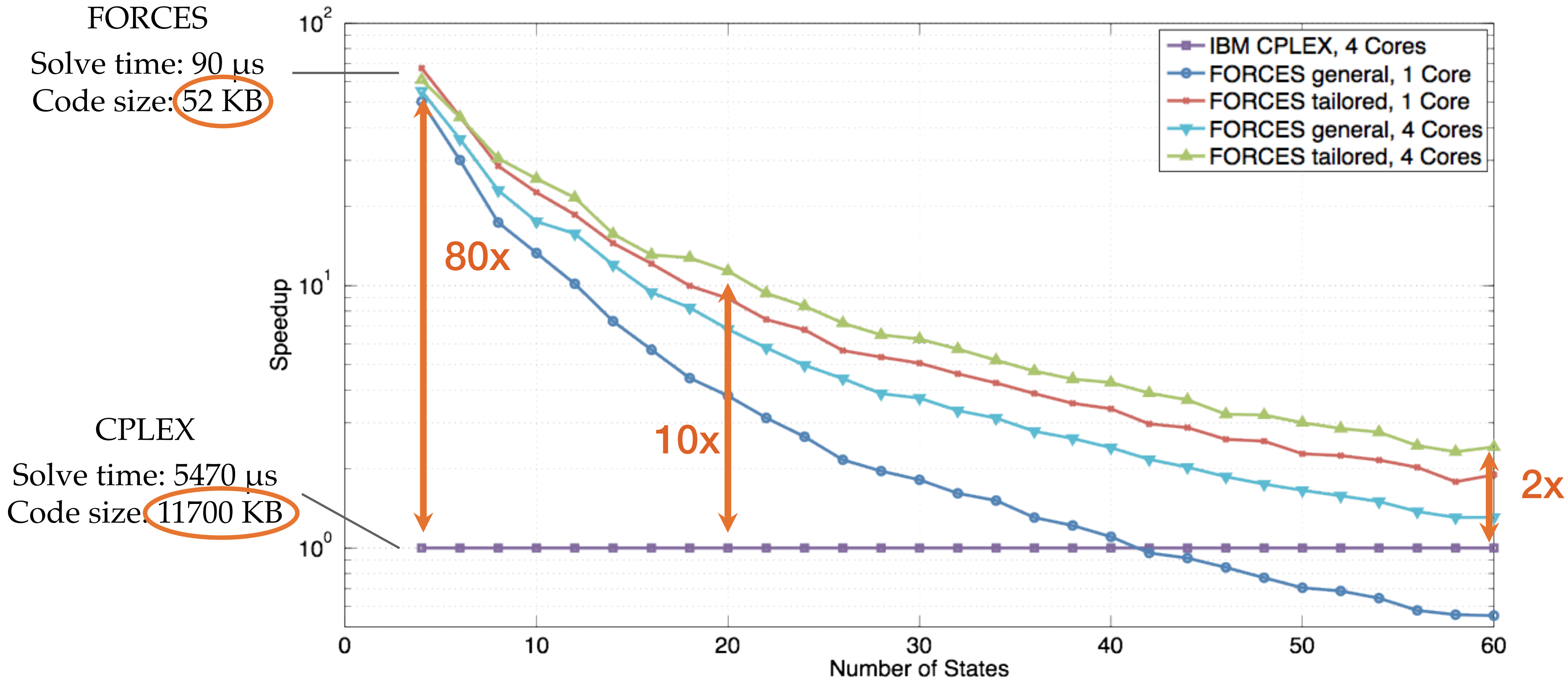
MATLAB MEX interface for rapid prototyping



# Speedups Compared to IBM's CPLEX

- Standard MPC problem for oscillating chain of masses (on Intel i5 @3.1 GHz)
- CPLEX N / A on embedded systems



# Some Early Users of FORCES

KATHOLIEKE UNIVERSITEIT LEUVEN

Nonlinear MPC & MHE with ACADO

*Milan Vukov, KU Leuven, 2012*

ALSTOM

MPC for Wind Turbines

*Marc Guadayol, ALSTOM, 2012*

IDSC

Quadrotor Control

*Marc Müller, IDSC, ETH Zurich, 2012*

ABB

Adaptive MPC for Belt Drives

*Kim Listmann, ABB Ladenburg, 2012*

UNIVERSITY OF CAMBRIDGE

Danfoss

TUM TECHNISCHE UNIVERSITÄT MÜNCHEN

UNIVERSITÀ DI PAVIA

DTU

OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG

LMS ENGINEERING INNOVATION

EPFL ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

STANFORD UNIVERSITY

Berkeley UNIVERSITY OF CALIFORNIA

# Micro-scale Race Cars

- 1:43 scale cars – 106mm
- Top speed: **5 m/s**
  (774 km/h scale speed)
- Full differential steering
- Position-sensing: External vision
- 50 Hz sampling rate

Project goals:

1. Plan optimal path online in dynamic race environment
2. Demonstrate real-time control optimizing car performance
3. Beat all human opponents!

Challenges:

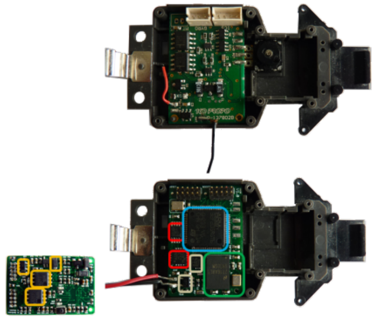
Interaction with multiple unpredictable opponents
Highly nonlinear dynamics
High-speed planning and control

# System Details

## Camera System

- Infrared spotlight
- Reflectors on cars
- 3.36 mm accuracy
- 100 Hz update rate at 1024 x 1200 pixels

## Embedded Board



- Custom built electronics
- Bluetooth communication
- IMUs & Gyro
- H-bridges for DC Motors
- ARM Cortex M4

## Tracks

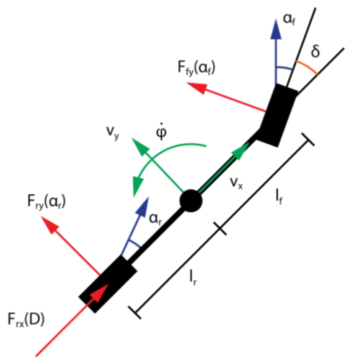
- Custom built high grip track

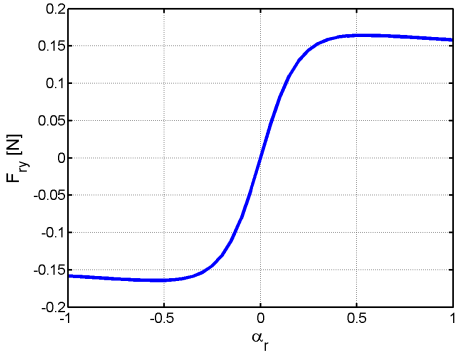
- Standard RCPtracks track

# Car Model and Model Analysis

## Car Model

Bicycle model with nonlinear lateral tire friction laws using a simplified Pacejka tire model:




$$\dot{x} = v_x \cos\varphi - v_y \sin\varphi$$

$$\dot{y} = v_x \sin\varphi + v_y \cos\varphi$$

$$\dot{\varphi} = \omega$$

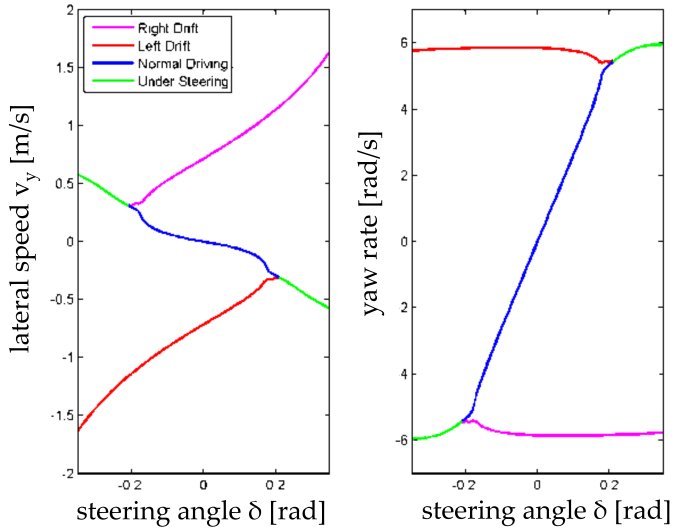
$$\dot{v}_x = \frac{1}{m}\left(F_{rx}(D) - F_{fy}\sin\delta + m v_y \dot{\varphi}\right)$$

$$\dot{v}_y = \frac{1}{m}\left(F_{ry}(\alpha_r) + F_{fy}\cos\delta - m v_x \dot{\varphi}\right)$$

$$\dot{\omega} = \frac{1}{J_z}\left(-F_{ry} l_r + F_{fy}\cos\delta l_r\right)$$

## Model Analysis

Analysis of zero accelerations (for fixed velocities) yield bifurcation:



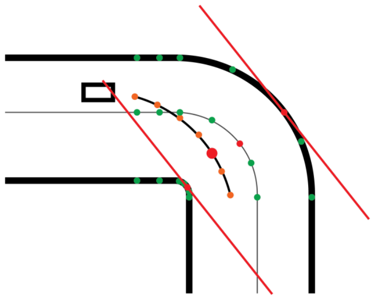

$$\delta = \text{const}$$
$$v_x = \text{const}$$
$$\dot{\omega} = 0$$
$$\dot{v}_y = 0$$

Stationary velocities for $v_x = 1.5$

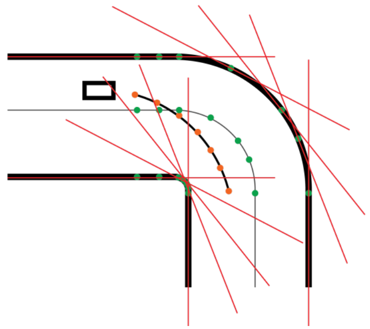
Stationary trajectories are circles, where the radius is related to the forward velocity and the yaw rate

# Path Planner and MPC

- Trajectory tracking MPC formulation:
  - re-linearizing the dynamics around trajectory
  - linear constraints to stay on track / avoid obstacles
  - 6 states, 2 inputs, horizon 16



$$
\begin{aligned}
\min \ & \sum_{i=0}^{N-1} (x_i - x_i^{\text{ref}})^T Q (x_i - x_i^{\text{ref}}) + (u_i - u_i^{\text{ref}})^T R (u_i - u_i^{\text{ref}}) + (x_N - x_N^{\text{ref}})^T P (x_N - x_N^{\text{ref}}) \\
\text{s.t. } & x_0 = x \\
& x_{i+1} = A_i x_i + B_i u_i + g_i \\
& G_i x_i \leq h_i \\
& (x_i, u_i) \in \mathbb{X} \times \mathbb{U}
\end{aligned}
$$



- Computation times on Intel Core i7 CPU @2.5 GHz:

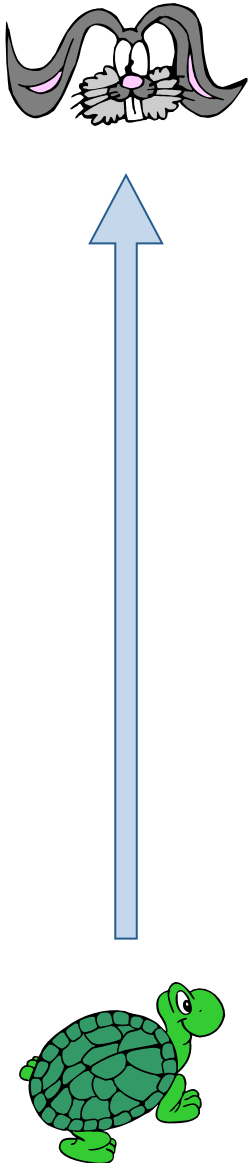
| | |
|---|---|
| Reference trajectory generation | 4.13 ms |
| Problem data generation | 1.19 ms |
| Solving QP with CPLEX | 6.00 ms |
| Solving QP with FORCES | 0.83 ms |

3.3 ms on
ARM Cortex
A9 @1.7 GHz

# Autonomous RC Racing Using FORCES

- Reference tracking MPC solved in 3.3 ms on ARM Cortex A9 based chip 

# Applications by the Automatic Control Lab

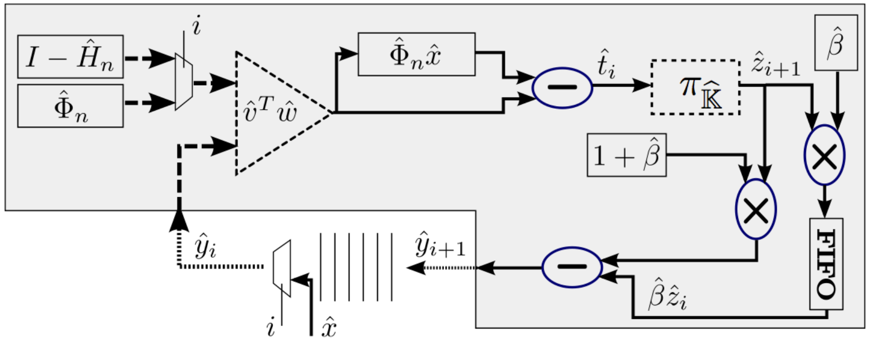


| | |
|---|---|
| 18 ns | Multi-core thermal management (EPFL) *[Zanini et al 2010]* |
| 10 $\mu$s | Voltage source inverters *[Mariethoz et al 2008]* |
| 20 $\mu$s | DC/DC converters (STM) *[Mariethoz et al 2008]* |
| 25 $\mu$s | Direct torque control (ABB) *[Papafotiou 2007]* |
| 50 $\mu$s | AC / DC converters *[Richter et al 2010]* |
| 5 ms | Electronic throttle control (Ford) *[Vasak et al 2006]* |
| 20 ms | Traction control (Ford) *[Borrelli et al 2001]* |
| 40 ms | Micro-scale race cars |
| 50 ms | Autonomous vehicle steering (Ford) *[Besselmann et al 2008]* |
| 500 ms | Energy efficient building control (Siemens) *[Oldewurtel et al 2010]* |

# Online MPC at Megahertz Rates using FPGAs

Hardware architectures for fast-gradient and ADMM methods parameterized by:

- Degree of parallelism
- Fixed-point computing precision

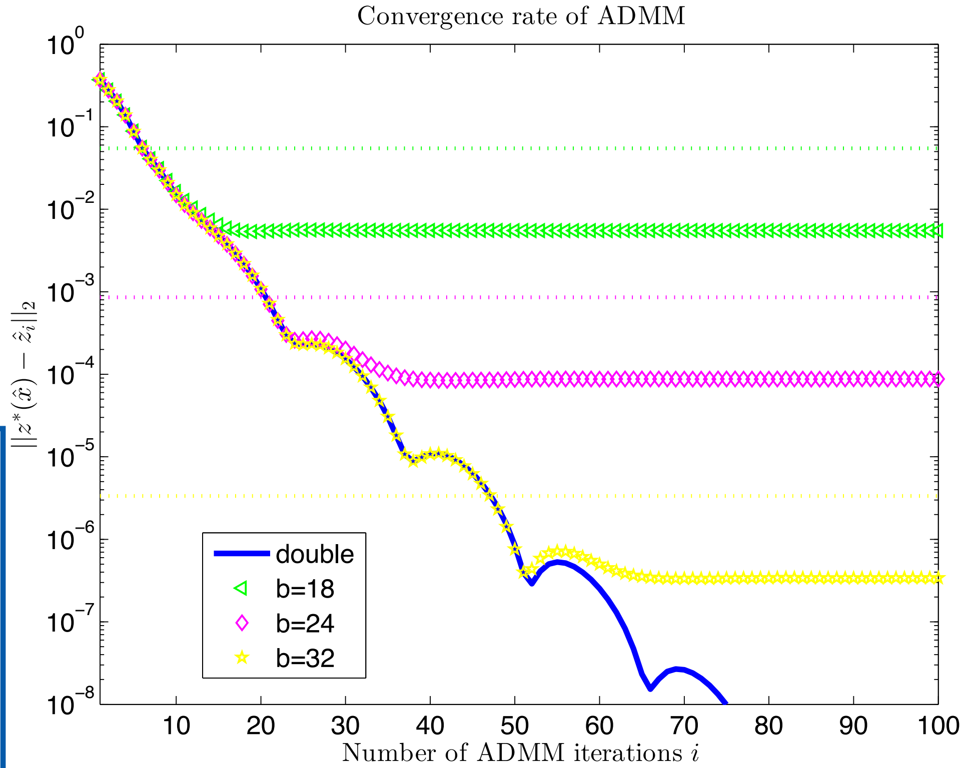



~250 variable optimization problem:

| State-of-the-art: | FPGAs: |
|---|---|
| ✦ ~5 ms | ✦ ~0.5 µs |
| ✦ Desktop @ 3GHz | ✦ Embedded @ 400MHz |
| ✦ ~70 Watts | ✦ ~5 Watts |

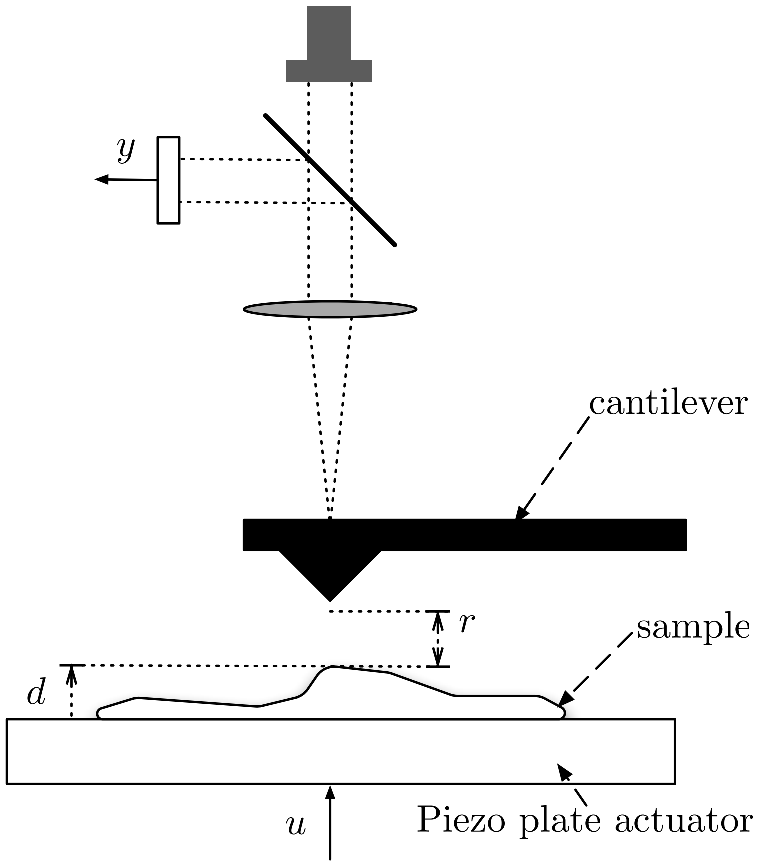
Fixed-point arithmetic analysis:

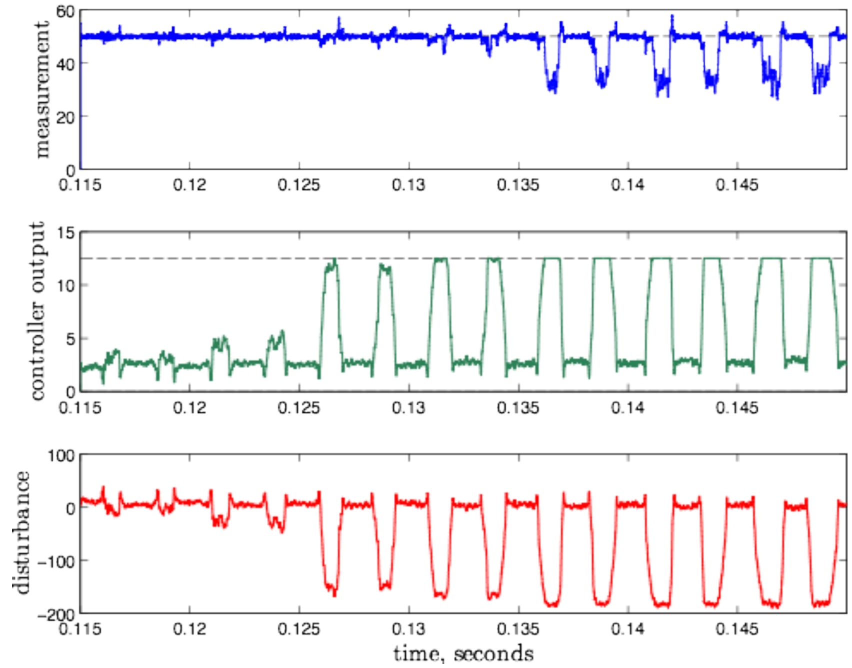
- ensures reliable operation
- Solution accuracy specs ⟶ # bits



# Online MPC at Megahertz Rates using FPGAs

MPC of an Atomic Force Microscope, in collaboration with IBM




✓ Same tracking performance as double precision state-of-the-art solver

✓ 700 kilohertz on 1 Watt FPGA

✓ >1 megahertz on high-performance FPGA

Jerez, Goulart, Richter, Constantinides, Kerrigan, Morari
*Embedded Predictive Control on an FPGA using the Fast Gradient Method"*, in ECC 2013

# Conclusions

- Computation technology is not limiting the application of MPC at any speed for any size problem
- When and where to employ MPC in industry is still a matter of judgment (modeling, maintenance, robustness)

Thank you, Keith!