Imperial College London

BUSINESS SCHOOL

Dynamic Portfolio Optimisation with Trading Costs

James Sefton and Sylvain Champonnois

Glover Celebration

Cambridge September 2013

Problem Motivation

- Quantitative Investing has had to respond to the crisis
  - Introduce faster signals (news, short term reversals)
  - To combine signals of different horizons (different decay rates)
  - To explicitly control trade costs (often the difference between funds)
  - To rebalance more frequently – maybe continuously

- All previous attempts (e.g BGI, AHL) at dynamic portfolio construction were *ad hoc* and eventually scrapped because
  - Lacked Transparency (portfolio positions must all be justifiable)
  - Could not cope with universe of 500+ assets (computation time)
  - Did not have an intuitive structure (trade-offs need to be clear)

Contributions of work

1. There exists a unique price of risk that is a function of the investor holdings
2. Strategy is to trade (in proportion to costs) to a target portfolio
3. Target portfolio has a clear structure
   - It is a weighted average of instantaneous mean-variance portfolios defined in terms of the price of risk
   - Weights are a function of trading costs and the discount rate only
4. Risk aversion introduces (as well as within period diversification)
   - Intertemporal hedging motives à la Merton
   - Distrust costly assets whose position is sensitive to volatile future opportunities

Our Stylised Model

Imperial College London BUSINESS SCHOOL

- There is a risk free asset that delivers an instantaneous return *r*
- Let $\boldsymbol{p_t}$ be the price of the *k* risky assets with returns

$$d\boldsymbol{p}_t = \left(\mu + C_s \boldsymbol{s}_t + r\boldsymbol{p}_t\right)dt + d\varepsilon_{p,t}$$

Risk Premiums are a function of economic states

where $\boldsymbol{s_t}$ are the economic states and evolve

$$d\boldsymbol{s}_t = A_s \boldsymbol{s}_t + d\varepsilon_{s,t} \qquad \text{and} \qquad \begin{bmatrix} \varepsilon_{s,t} \\ \varepsilon_{p,t} \end{bmatrix} \sim N\left(0, \Sigma\right)$$

Innovations to forecasts correlated with returns – Merton hedging

- Investor holds $\boldsymbol{n_t}$ shares of each asset and therefore has wealth $w_t = \boldsymbol{n}_t^T \boldsymbol{p}_t$. The she must choose her trading strategy, $\boldsymbol{\tau_t}$,

$$d\boldsymbol{n}_t = \tau_t dt \qquad \text{which costs} \qquad \frac{\tau_t^T \Lambda \tau_t}{2} dt$$

and consumption c_t.

Costs are quadratic -=> impact costs

The Objective

- The states of the problem are the exogenous economic states, $\boldsymbol{s}_t$, the current portfolio holdings, $\boldsymbol{n}_t$, and wealth w_t which evolves

$$dw_t = \left(rw_t - c_t + \boldsymbol{n}_t^T C_s \boldsymbol{s}_t - \frac{\boldsymbol{\tau}_t^T \Lambda \boldsymbol{\tau}_t}{2} \right) dt + \boldsymbol{n}_t^T \boldsymbol{\varepsilon}_{p,t}$$

- Our investor chooses trading intensity $\boldsymbol{\tau}_t$ and consumption c_t so as to maximise her expected discounted utility

$$J = \max_{\tau_t, c_t} E\left[\int_0^\infty e^{-\delta s} \left(-e^{-\beta c_s} \right) ds \right]$$

subject to the transversality condition that $\lim_{t \to \infty} E\left(e^{-\delta t} w_t \right) = 0$

.. and then as a control problem

- The problem states are $\boldsymbol{x}_t = \begin{bmatrix} \boldsymbol{s}_t \\ \boldsymbol{n}_t \end{bmatrix}$ which evolve

$$\begin{bmatrix} \varepsilon_{s,t} \\ \varepsilon_{p,t} \end{bmatrix} = \varepsilon_t$$

$$d\boldsymbol{x}_t = \underbrace{\begin{bmatrix} A_s & 0 \\ 0 & 0 \end{bmatrix}}_{A} \boldsymbol{x}_t dt + \underbrace{\begin{bmatrix} A_s & 0 \\ 0 & 0 \end{bmatrix}}_{B_1} d\varepsilon_t + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{B_2} \tau_t dt$$

- Wealth evolves

$$dw_t = \left(rw_t - c_t + \frac{1}{2} \boldsymbol{x}_t^T \underbrace{\begin{bmatrix} C_s^T & 0 \\ 0 & I \end{bmatrix}}_{C^T} \underbrace{\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} C_s & 0 \\ 0 & I \end{bmatrix}}_{C} \boldsymbol{x}_t - \frac{\tau_t^T \Lambda \tau_t}{2} \right) dt \ldots$$

$$\ldots + \boldsymbol{x}_t^T \underbrace{\begin{bmatrix} C_s^T & 0 \\ 0 & I \end{bmatrix}}_{C^T} \underbrace{\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}}_{D} d\varepsilon_t$$

- The control variables are τ_t and c_t chosen to maximise utility.

The Solution Procedure

Imperial College London BUSINESS SCHOOL

1. The problem is now phrased as a standard risk sensitive control problem.

2. We can write down the Hamilton-Jacobi-Bellman (HJB) equation.

3. Assume that the value function is quadratic function of the states $\boldsymbol{x_t}$

$$J = K \exp\left(-\delta t - \gamma\left[w_t + \tfrac{1}{2}\boldsymbol{x}_t^T \Pi \boldsymbol{x}_t\right]\right)$$

and substitute into the HJB equation

4. Show that this solves the HJB if Π satisfies a Riccatti equation.

The Solution Procedure

Imperial College London BUSINESS SCHOOL

- *Theorem:* Suppose that the Hamiltonian

$$H = \begin{bmatrix} \left(A - rI - \gamma B_1 \Sigma D^T RC\right) & B_2 \Lambda^{-1} B_2^T - \gamma B_1 \Sigma B_1^T \\ -C^T \left(R - \gamma RD\Sigma D^T R\right) C & -\left(A - rI - \gamma B_1 \Sigma D^T RC\right)^T \end{bmatrix}$$

is $H \in dom(Ric)$. Let $\Pi = Ric(H)$ is the stabilising solution and is partitioned conformally with the states x.

$$\Pi = \begin{bmatrix} \Pi_{ss} & \Pi_{ns}{}^T \\ \Pi_{ns} & \Pi_{nn} \end{bmatrix}$$

Then the optimal trading intensity τ_t and consumption c_t

$$\tau_t = \Lambda^{-1} B_2^T \Pi x_t$$

$$c_t = r \underbrace{\left(w_t + \tfrac{1}{2} \boldsymbol{x}_t^T \Pi \boldsymbol{x}_t\right)}_{\text{Wealth + PV of future net resources}} + \frac{\delta - r}{\gamma} + \tfrac{1}{2} trace\left(B_1 \Sigma B_1^T \Pi\right)$$

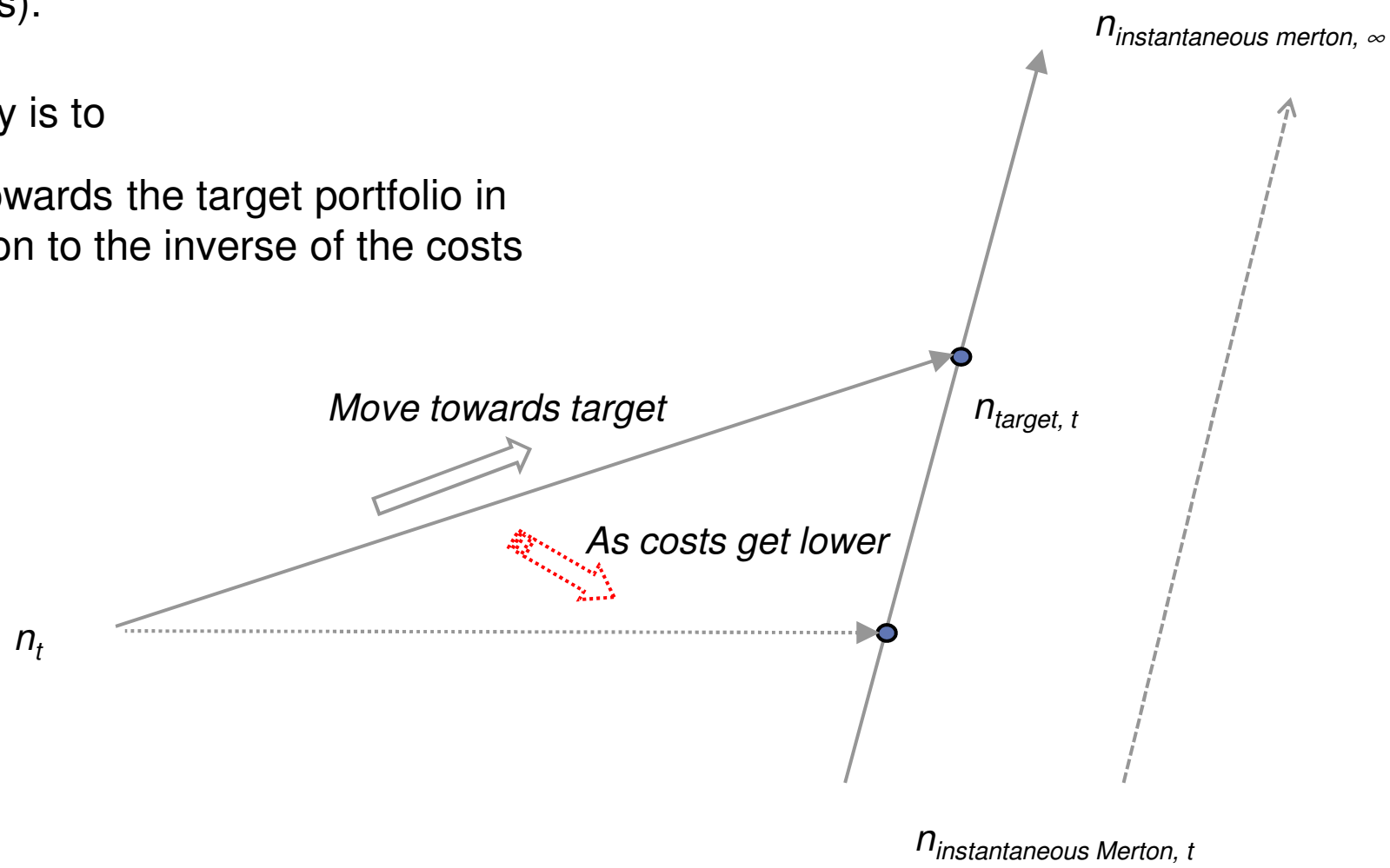
where $\gamma = r\beta$ exists if $\Pi_{nn} < 0$ and $\Pi_{ss} > 0$.

The Structure of the Optimal Portfolio

1. The target portfolio is a weighted average of expected instantaneous mean variance portfolio (Merton portfolios).

The strategy is to

1. Trade towards the target portfolio in proportion to the inverse of the costs



The optimal trading rule

Imperial College London BUSINESS SCHOOL

- Define the target portfolio as the portfolio that maximises the present value of future resources i.e

$$\max_{n_t} \boldsymbol{x}_t^T \Pi \boldsymbol{x}_t$$

then the trading rule can be written

$$\tau_t = -\Lambda^{-1} B_2^T \Pi B_2 \left(n_t^{\text{Target}} - n_t \right)$$

The Instantaneous Mean Variance Portfolio

Imperial College London BUSINESS SCHOOL

- Define the Merton Instantaneous Portfolio as the portfolio that maximises the instantaneous mean variance problem

$$\max_{n_t}\left(-\frac{1}{\gamma}\exp\left(-\gamma\left(\underbrace{n_t^T\cdot dp_t}_{\text{Instantaneous Return}}+\underbrace{x_t^T\begin{bmatrix}\Pi_{ss}\\ \Pi_{ns}\end{bmatrix}\sigma_s d\varepsilon_t}_{\text{Innovation to the future opportunity set}}\right)\right)\right)$$

- Or the equivalent quadratic problem

$$\max_{n_t}\left(\underbrace{\boldsymbol{x}_t^T C^T RC\boldsymbol{x}_t}_{\text{Expected return to Portfolio}}-\gamma\underbrace{\boldsymbol{x}_t^T\left(\Pi B_1+C^T RD\right)\Sigma\left(\Pi B_1+C^T RD\right)^T}_{\text{Adjusted Instantaneous Variance of Net Resources}}\boldsymbol{x}_t\right)$$

The Target Portfolio

Imperial College London BUSINESS SCHOOL

- The optimal target portfolio is weighted average of future Merton Instantaneous Portfolios

$$n_t^{\text{Target}} = E\left(\int_0^\infty e^{-rt} H_t n_t^{\text{Instant Merton}} dt\right) \Big/ \left(\int_0^\infty e^{-rt} H_t dt\right)$$

$$\text{where } \mathrm{H}_t = \exp\left(\Lambda^{-1} B_2^T \Pi B_2 t\right)$$

And the link to H∞

- Definition: An equivalent probability measure Q to P is said to belong to the set Q(ω) if there exists a Radon-Nikodym derivative q_t where

$$\log q_t = -\tfrac{1}{2}\int_0^t \theta_\upsilon^T \Sigma \theta_\upsilon d\upsilon + \int_0^t \theta_\upsilon^T \sigma d\varepsilon$$

for some $\theta \in H_2$.

- Then the consumption path that solves our earlier problem solves

$$\sup_{c_t} \inf_{q_t \in Q(\omega)} E^Q\left(\int_0^\infty e^{-rt} c_t dt\right) + \tfrac{1}{2\gamma} E^Q\left(\int_0^\infty e^{-rt} \log q_t dt\right)$$

over all admissible consumption paths (i.e. budget feasible). Further the change of measure (price of risk) supporting this path is

$$\theta_t = -\gamma\sigma^T\left(\Pi B_1 + C^T RD\right) x_t$$

An example in discrete time

Timing a Value Tilt

Model: Expected Returns to a Long-Short Value Strategy

1. r_{t+1} - Returns to top 3rd minus bottom 3rd stocks sorted by book to price.
2. S_{t+1} - Difference in logs of book to price of top 3rd minus bottom 3rd.
3. M_{t+1} - A weighted average of lagged returns r_{t+1}

State Equations

$$\begin{cases} \left(S_{t+1} - \bar{S}\right) = \varphi_S\left(S_t - \bar{S}\right) + \eta_t \\ M_{t+1} = r_{t+1} + \varphi_M M_t \end{cases}$$

Output Equation

$$r_{t+1} = \mu + \alpha_S\left(S_t - \bar{S}\right) + \alpha_M M_t + \varepsilon_t$$

$$Cov\left(\begin{bmatrix} \varepsilon_t \\ \eta_t \end{bmatrix}\right) = \begin{bmatrix} \sigma_\varepsilon^2 & -\rho\sigma_\varepsilon\sigma_\eta \\ -\rho\sigma_\varepsilon\sigma_\eta & \sigma_\eta^2 \end{bmatrix}$$

- $\eta < 0$ implies lower future returns. However as Cov(η, ε)< 0 , hedge by holding ***more*** of value portfolio.
- $\varepsilon < 0$ implies lower future returns due to momentum. However can hedge by holding ***less*** of value portfolio.
- Mean reversion leads to positive, momentum to negative, hedging demands

Estimate the model by maximum likelihood

Imperial College London BUSINESS SCHOOL

1. r_{t+1} - Returns to top 3rd minus bottom 3rd stocks sorted by book to price.
2. S_{t+1} - Difference in logs of book to price of top 3rd minus bottom 3rd.
3. M_{t+1} - A weighted average of lagged returns r_{t+1}

State Equations

$$\begin{cases} (S_{t+1} - 1.45) = \underset{(0.017)}{0.968}(S_t - 1.45) + \eta_t \\ M_{t+1} = r_{t+1} + \underset{(0.13)}{0.75} M_t \end{cases}$$

Output Equation

$$r_{t+1} = \underset{(0.0007)}{0.0018} + \underset{(0.011)}{0.028}(S_t - 1.45) + \underset{(0.032)}{0.132}(M_t - 0.008) + \varepsilon_t$$

$$Cov\left(\begin{bmatrix} \varepsilon_t \\ \eta_t \end{bmatrix}\right) = \begin{bmatrix} \mathbf{11.1\%} & -0.43 \\ & \mathbf{23.5\%} \end{bmatrix}$$

Diagonal- Annualised Volatility

Off Diagonal - Correlations

- Estimated on monthly data from Jan 1994 – April 2012
- The half-life of the momentum signal is just over 3 months
- There is strong negative correlation between the innovations to returns and innovations to the opportunity set.
- The R^2 of the output equation is 0.12

The Investment Objective

Imperial College London BUSINESS SCHOOL

- The investor chooses his tilt to value. He is benchmarked to the market. Thus his fully invested portfolio returns are:

$$r_{t+1}^{Mkt} + w_t r_{t+1}$$

We assume a constant return to the market of 4% per year, and covariance

$$Cov\left(\begin{bmatrix} \eta_t \\ \varepsilon_t \\ \varepsilon_t^{Mkt} \end{bmatrix}\right) = \begin{bmatrix} \mathbf{23.5\%} & -0.43 & -0.15 \\ & \mathbf{11.1\%} & 0.33 \\ & & \mathbf{20\%} \end{bmatrix}$$

Diagonal- Annualised Volatility

Off Diagonal - Correlations

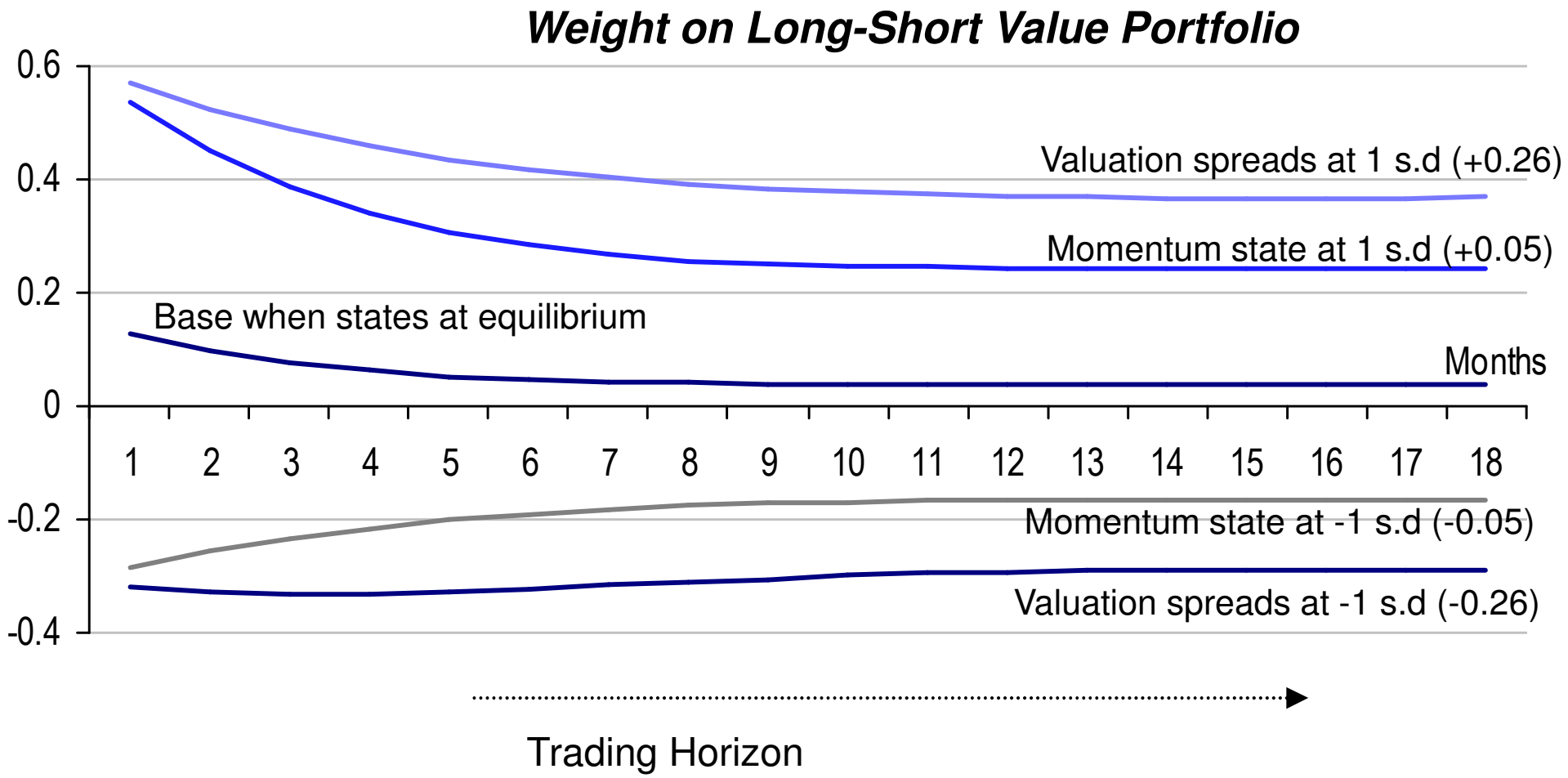
- He dynamically adjust his tilts to maximise the CRA of his wealth after *T* periods

- Costs are calibrated so that a 1% increase in the value tilt in a month costs 0.1bps, whereas a 10% increase costs 10bps.

Target Portfolio

Imperial College London BUSINESS SCHOOL

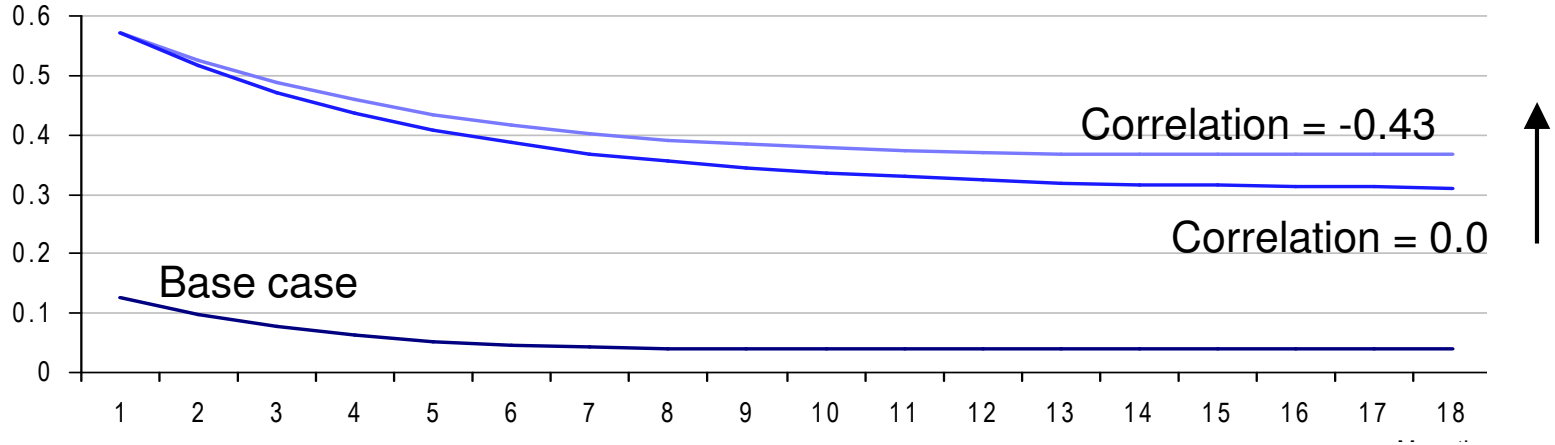
- Longer the horizon the lower the weight on value, due to anticipated trading costs.
- Trading momentum is more expensive, so allocation fall faster with horizon.



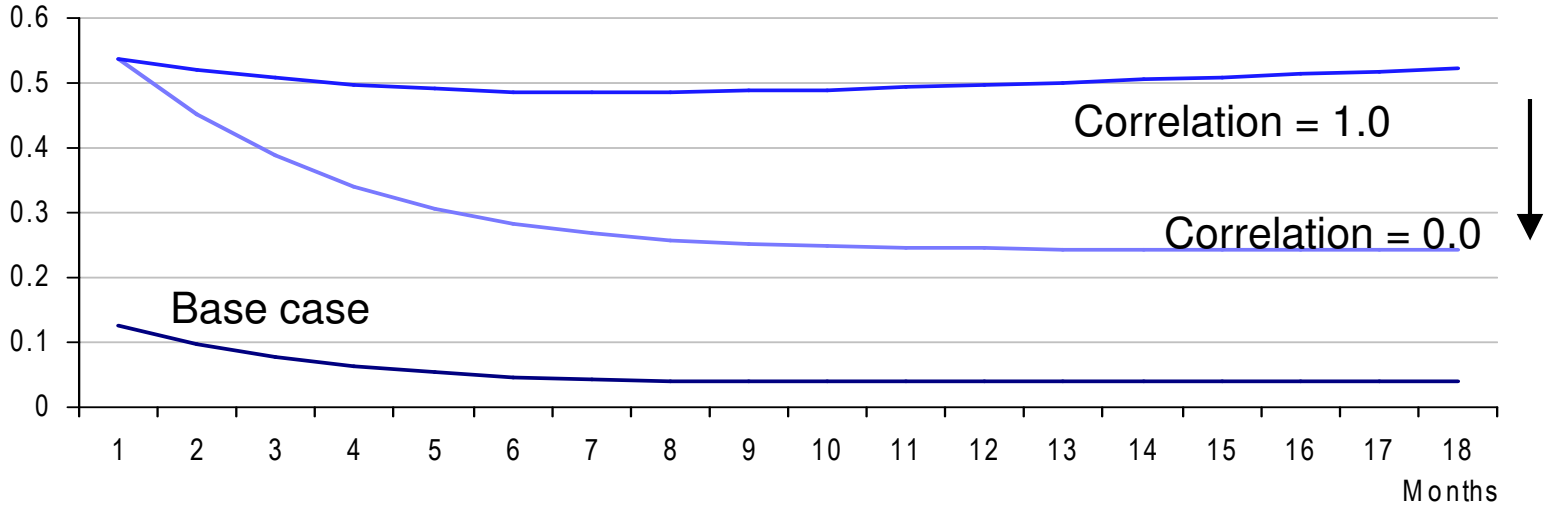
Impact of Intertemporal Hedging Motives on Target Portfolio

Imperial College London BUSINESS SCHOOL

- Weight on value if the ***Valuation Spread*** is +1 s.d (0.26) above mean.
- Over a long horizon, intertemporal hedging increases weight on value.



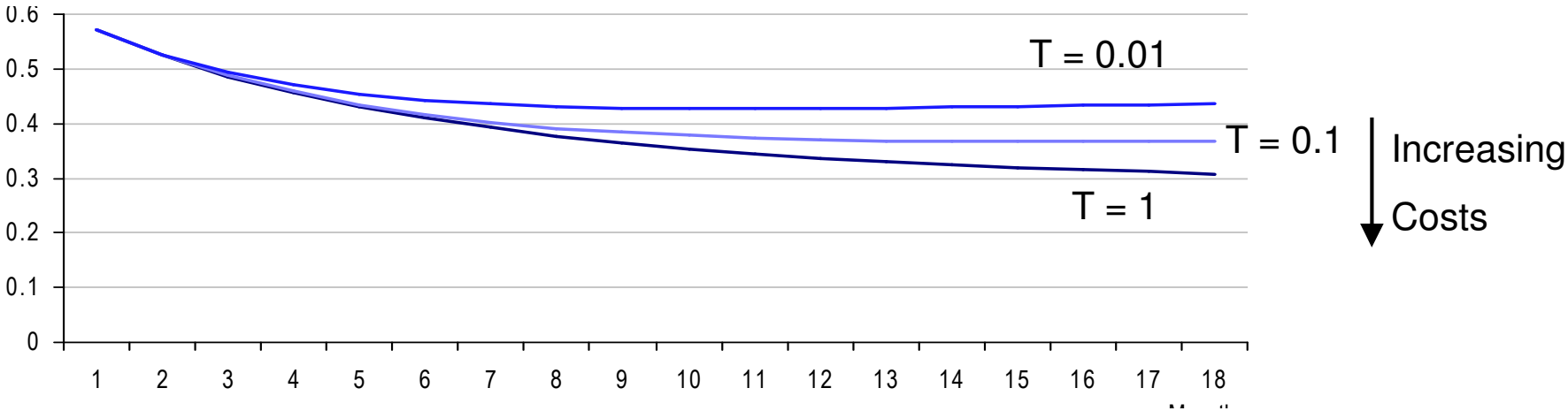
- Weight on value if the ***Momentum state*** is +1 s.d (0.05) above mean.
- Over a long horizon, intertemporal hedging significantly reduces weight on value.



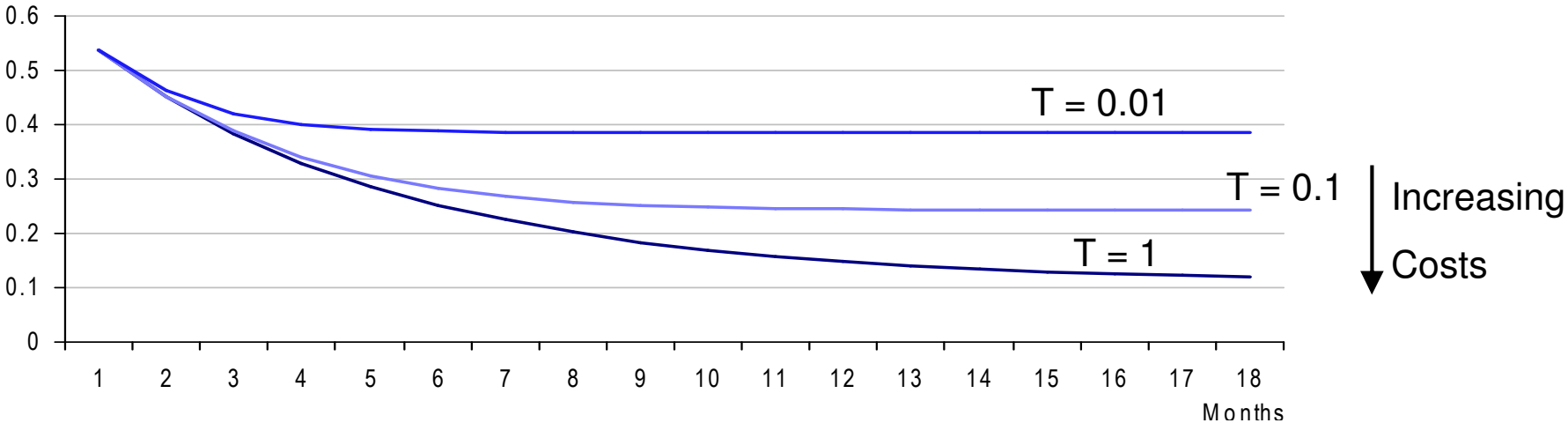
Impact of Costs on Target Portfolio

Imperial College London BUSINESS SCHOOL

- Weight on value if the ***Valuation Spread*** is +1 s.d (0.26) above mean.
- Over a long horizon, costs gently reduce the weight on value



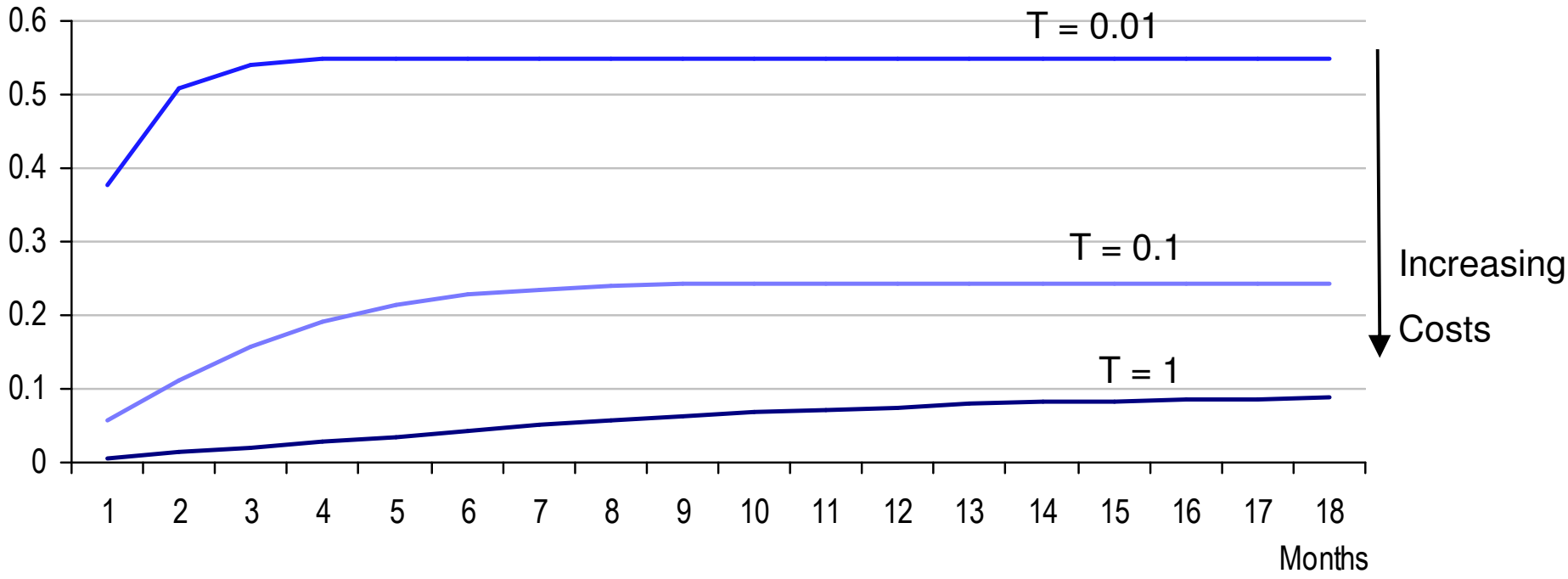
- Weight on value if the ***Momentum state*** is +1 s.d (0.05) above mean.
- Over a long horizon, costs *significantly* reduce the weight on momentum.



Impact of Costs on Trading Speed

Imperial College London BUSINESS SCHOOL

- Trading Speeds increase as horizon increases – as the accruing benefit increases
- Trading Speeds reduce significantly as costs increases (perhaps not very surprising!)



Conclusions

Imperial College London BUSINESS SCHOOL

When comparing single period optimisation to a dynamic optimisation

1. Not accounting for intertemporal hedging will
   1. Seriously overweight Momentum or trend following strategies
   2. Underweight value strategies

2. Not accounting for trading costs in relation to horizon of the strategy
   1. Overweights momentum again far more heavily than a value strategy

3. Suggests a separation strategy:
   1. Estimate the optimal long run target portfolio on the basis of costs and hedging demands
   2. Estimate the trading speed at the second stage based on maximizing net returns.

Part 3: Appendix on Trading Costs

Transactions Costs – Linear in the size of trade

Imperial College London BUSINESS SCHOOL

- Leland (1996, 1999) and Atkinson (2004, 2010) analysed optimal rebalancing when costs are proportional to change in portfolio weights, $TC = k|\Delta w|$
- Reasonable model if trades are *small*
  - Cost is of trading a block of shares is

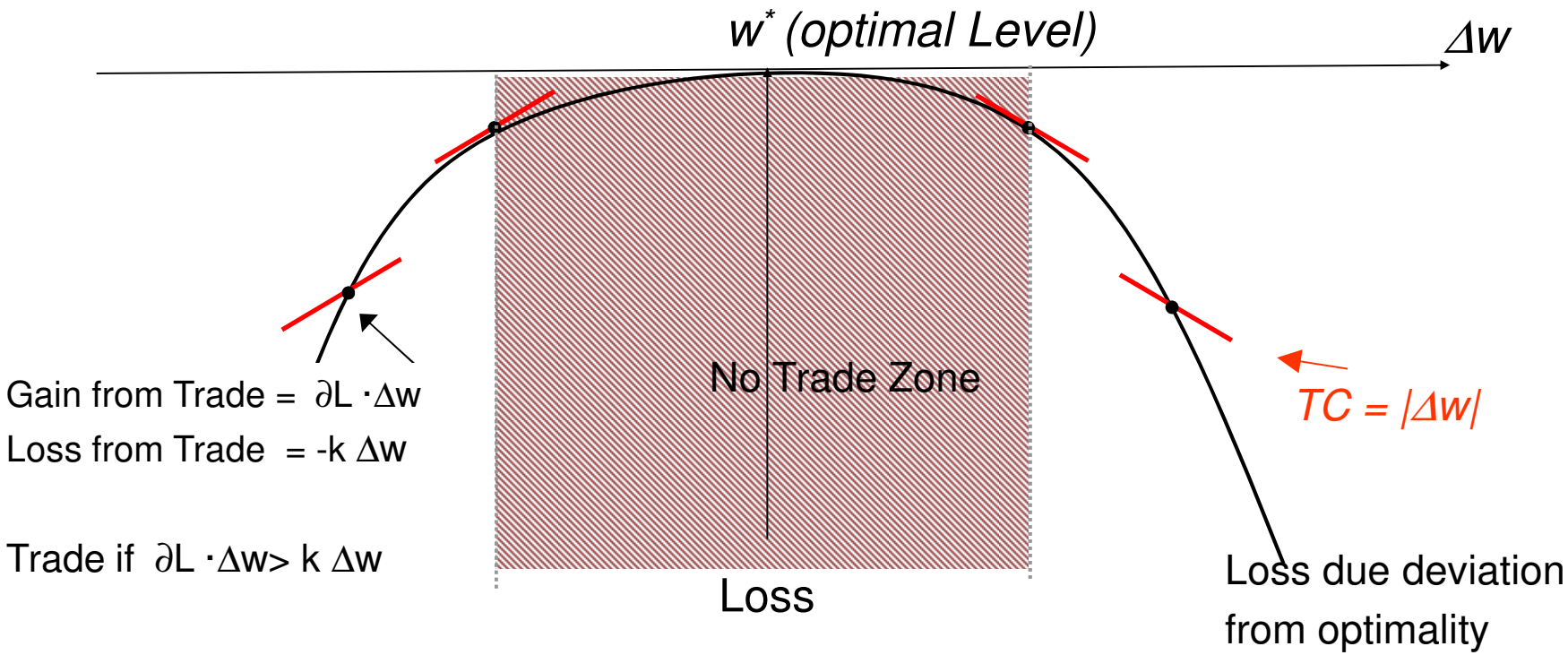
$$\tfrac{1}{2}(\text{Bid-Ask Spread}) \times (\text{Number of Shares}) \Rightarrow$$

$$k = \tfrac{1}{2}(\text{Bid-Ask Spread}) / (\text{Price of Share})$$

- This model of costs implies no-trade zone.
  - In zone – no trade
  - Outside – trade back to edge of the zone

Transactions Costs – No trade-zone

- Intuition is easy:
  - Costs from a non optimal portfolio will be convex in deviation from optimality
  - Cost of trading are linear in the size of the trade



- Capital gains can be analysed similarly (effectively just another cost)
- In multi-asset problem, numerically difficult to find borders of no trade-zone.

Transaction Costs – Large Trades

Imperial College London BUSINESS SCHOOL

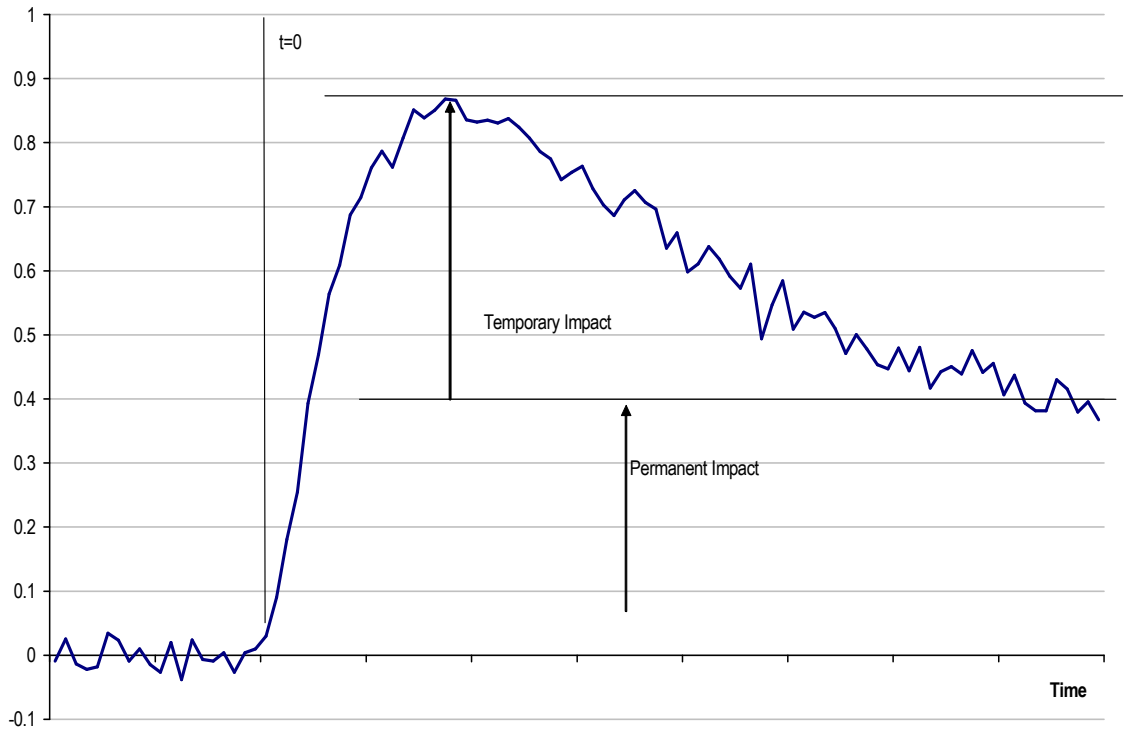
- Trading costs can be divided into:
  1. Explicit Costs: Order processing costs (compensated through the bid-ask spread)
  2. Implicit Costs : Inventory and informational costs. These cause prices to move to compensate the intermediary for risk of a large book and being the wrong side of an informed trade respectively.

- Madhaven (2000) states that

  'while most researchers recognize that quoted spreads are small, implicit trading costs can actually be economically significant because large trades move prices'.

- And similarly, Almgren (2005)

  'For large trades, the most important component of these is the impact of the trader's own actions on the market. These costs are notoriously difficult to measure.'



The Assumption of Quadratic Transaction Costs

Imperial College London BUSINESS SCHOOL

- A block trade Δw_t is split into *n* smaller trades, with each of these trades executed at prices, p_{t+i} for *i=0,1..n-1* , where

$$p_t - \tilde{p}_t = \tfrac{1}{2} S \operatorname{sgn}(\Delta w_t^i)$$

$$\Delta \tilde{p}_{t+1} = \alpha \tfrac{1}{2} S \operatorname{sgn}(\Delta w_t^i) + \varepsilon_t$$

$\tilde{p}_t$ is mid price and S is the bid-ask the spread. The coefficient α is the information transfer of the trade. Stoll and Huang (97) and Stoll (00) estimate α to be between 5-45%.

- Aggregate over the n trades to find the cost of trading the block.

$$TC = \sum_{i=0}^{i=n-1} p_{t+i} - n\tilde{p}_t = \frac{nS}{2}\left(\left(1 - \tfrac{\alpha}{2}\right) + \frac{\alpha n}{2}\right) + noise$$

So if n is large and α>0, cost is approximately quadratic in the size of the block, *n.*

What the research says:

Imperial College London BUSINESS SCHOOL

- Almgren (2005) is the only work on large dataset with details of breakdown of orders to individual trades. He finds

$$TC = \underbrace{k_1 \left(\Delta w_t\right)^2}_{\substack{\text{Permanent}\\ \text{Impact}}} + \underbrace{k_2 \left(\Delta w_t\right)^{1.6}}_{\substack{\text{Temporary}\\ \text{Impact}}}$$

where k_1 and k_2 are function of stock's volatility, liquidity, average daily volume and participation rate.

- Kissel *et al* (2005, 2008), I-star model:

$$TC = \underbrace{k_1 \left(\Delta w_t\right)^{1.36}}_{\substack{\text{Permanent}\\ \text{Impact}}} + \underbrace{k_2 \left(\Delta w_t\right)^{2.05}}_{\substack{\text{Temporary}\\ \text{Impact}}}$$

where again k_1 and k_2 are function of stock's volatility, liquidity and average daily volume.

Conclusion

Quadratic reasonable assumption for large trades - but channel is through the price impact

References

- Doyle, John C., et al. "State-space solutions to standard H2 and H∞ control problems." *Automatic Control, IEEE Transactions on* 34.8 (1989): 831-847.
- Fleming, Wendell H., and William M. McEneaney. "Risk-sensitive control on an infinite time horizon." *SIAM Journal on Control and Optimization* 33.6 (1995): 1881-1915.
- Fleming, W. H., and S. J. Sheu. "Risk-sensitive control and an optimal investment model II." *Annals of Applied Probability* (2002): 730-767.
- Fleming, W. H., and S. J. Sheu. "Risk-Sensitive Control and an Optimal Investment Model." *Mathematical Finance* 10.2 (2000): 197-213.
- Garleanu, Nicolae B., and Lasse H. Pedersen. *Dynamic trading with predictable returns and transaction costs*. No. w15205. National Bureau of Economic Research, 2009.
- Muirhead, R. J. (1982) “Aspects of Multivariate Statistical Theory” John Wiley
- Whittle, Peter. *Risk-sensitive optimal control*. New York: Wiley, 1990.
- Whittle, Peter. "Risk sensitivity, a strangely pervasive concept."*Macroeconomic Dynamics* 6.1 (2002): 5-18.
- Zhou, Kemin, John Comstock Doyle, and Keith Glover. *Robust and optimal control*. Vol. 40. New Jersey: Prentice Hall, 1996.
- Zhou, Xun Yu, and Duan Li. "Continuous-time mean-variance portfolio selection: A stochastic LQ framework." *Applied Mathematics and Optimization* 42.1 (2000): 19-33.