Dynamic Portfolio Optimisation with Trading Costs

James Sefton and Sylvain Champonnois

Glover Celebration
Cambridge September 2013
Problem Motivation

- Quantitative Investing has had to respond to the crisis
  - Introduce faster signals (news, short term reversals)
  - To combine signals of different horizons (different decay rates)
  - To explicitly control trade costs (often the difference between funds)
  - To rebalance more frequently – maybe continuously

- All previous attempts (e.g. BGI, AHL) at dynamic portfolio construction were *ad hoc* and eventually scrapped because
  - Lacked Transparency (portfolio positions must all be justifiable)
  - Could not cope with universe of 500+ assets (computation time)
  - Did not have an intuitive structure (trade-offs need to be clear)
Contributions of work

1. There exists a unique price of risk that is a function of the investor holdings

2. Strategy is to trade (in proportion to costs) to a target portfolio

3. Target portfolio has a clear structure
   – It is a weighted average of instantaneous mean-variance portfolios defined in terms of the price of risk
   – Weights are a function of trading costs and the discount rate only

4. Risk aversion introduces (as well as within period diversification)
   – Intertemporal hedging motives à la Merton
   – Distrust costly assets whose position is sensitive to volatile future opportunities
Our Stylised Model

• There is a risk free asset that delivers an instantaneous return $r$

• Let $p_t$ be the price of the $k$ risky assets with returns

\[ dp_t = (\mu + C_s s_t + rp_t) dt + d\varepsilon_{p,t} \]

where $s_t$ are the economic states and evolve

\[ ds_t = A_s s_t + d\varepsilon_{s,t} \quad \text{and} \quad \begin{bmatrix} \varepsilon_{s,t} \\ \varepsilon_{p,t} \end{bmatrix} \sim N(0, \Sigma) \]

• Investor holds $n_t$ shares of each asset and therefore has wealth $w_t = n_t^T p_t$. The she must choose her trading strategy, $\tau_t$,

\[ dn_t = \tau_t dt \quad \text{which costs} \quad \frac{\tau_t^T \Lambda \tau_t}{2} dt \]

and consumption $c_t$. 

Risk Premiums are a function of economic states
Innovations to forecasts correlated with returns – Merton hedging
Costs are quadratic => impact costs
The Objective

- The states of the problem are the exogenous economic states, $s_t$, the current portfolio holdings, $n_t$, and wealth $w_t$ which evolves

$$dw_t = \left( rw_t - c_t + n_t^T C_s s_t - \frac{\tau_t^T \Lambda \tau_t}{2} \right) dt + n_t^T \varepsilon_{p,t}$$

- Our investor chooses trading intensity $\tau_t$ and consumption $c_t$ so as to maximise her expected discounted utility

$$J = \max_{\tau_t, c_t} E \left[ \int_0^\infty e^{-\delta s} \left( -e^{-\beta c_s} \right) ds \right]$$

subject to the transversality condition that $\lim_{t \to \infty} E\left( e^{-\delta_t} w_t \right) = 0$
.. and then as a control problem

- The problem states are $x_t = \begin{bmatrix} s_t \\ n_t \end{bmatrix}$ which evolve

$$dx_t = \begin{bmatrix} A_s & 0 \\ 0 & 0 \end{bmatrix} x_t dt + \begin{bmatrix} A_s & 0 \\ 0 & 0 \end{bmatrix} d\varepsilon_t + \begin{bmatrix} 0 \\ I \end{bmatrix} \tau_t dt$$

- Wealth evolves

$$dw_t = \left( rw_t - c_t + \frac{1}{2} x_t^T \begin{bmatrix} C_s^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} C_s & 0 \\ 0 & I \end{bmatrix} x_t - \frac{\tau_t^T \Lambda \tau_t}{2} \right) dt ...$$

$$... + x_t^T \begin{bmatrix} C_s^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} d\varepsilon_t$$

- The control variables are $\tau_t$ and $c_t$ chosen to maximise utility.
The Solution Procedure

1. The problem is now phrased as a standard risk sensitive control problem.

2. We can write down the Hamilton-Jacobi-Bellman (HJB) equation.

3. Assume that the value function is quadratic function of the states $x_t$

   \[ J = K \exp \left( -\delta t - \gamma \left[ w_t + \frac{1}{2} x_t^T \Pi x_t \right] \right) \]

   and substitute into the HJB equation

4. Show that this solves the HJB if $\Pi$ satisfies a Riccatti equation.
The Solution Procedure

• **Theorem:** Suppose that the Hamiltonian

\[
H = \begin{bmatrix}
(A - rI - \gamma B_1 \Sigma D^T RC) & B_2 \Lambda^{-1} B_2^T - \gamma B_1 \Sigma B_1^T \\
-C^T (R - \gamma RD\Sigma D^T R)C & -(A - rI - \gamma B_1 \Sigma D^T RC)^T
\end{bmatrix}
\]

is \( H \in \text{dom}(\text{Ric}) \). Let \( \Pi = \text{Ric}(H) \) is the stabilising solution and is partitioned conformally with the states \( x \).

Then the optimal trading intensity \( \tau_t \) and consumption \( c_t \)

\[
\tau_t = \Lambda^{-1} B_2^T \Pi x_t
\]

\[
c_t = r \left( w_t + \frac{1}{2} x_t^T \Pi x_t \right) + \frac{\delta - r}{\gamma} + \frac{1}{2} \text{trace} \left( B_1 \Sigma B_1^T \Pi \right)
\]

Wealth + PV of future net resources

where \( \gamma = r\beta \) exists if \( \Pi_{nn} < 0 \) and \( \Pi_{ss} > 0 \).
The Structure of the Optimal Portfolio

1. The target portfolio is a weighted average of expected instantaneous mean variance portfolio (Merton portfolios).

The strategy is to

1. Trade towards the target portfolio in proportion to the inverse of the costs

As costs get lower

Move towards target
The optimal trading rule

- Define the target portfolio as the portfolio that maximises the present value of future resources i.e.

\[
\max_{n_t} x_t^T \Pi x_t
\]

then the trading rule can be written

\[
\tau_t = -\Lambda^{-1} B_2^T \Pi B_2 \left( n_t^{\text{Target}} - n_t \right)
\]
The Instantaneous Mean Variance Portfolio

- Define the Merton Instantaneous Portfolio as the portfolio that maximises the instantaneous mean variance problem

\[
\max_{n_t} \left\{ -\frac{1}{\gamma} \exp \left[ -\gamma \left( n_T \cdot d_p_t + x_T \left[ \Pi_{ss} \right] \sigma_s d\varepsilon_t \right) \right] \right\}
\]

- Or the equivalent quadratic problem

\[
\max_{n_t} \left\{ x_t^T C^T RCx_t - \gamma x_t^T \left( \Pi B_1 + C^T RD \right) \Sigma \left( \Pi B_1 + C^T RD \right)^T x_t \right\}
\]
The Target Portfolio

- The optimal target portfolio is weighted average of future Merton Instantaneous Portfolios

\[
n_t^{\text{Target}} = \frac{E \left( \int_0^\infty e^{-rt} H_t n_t^{\text{Instant Merton}} \, dt \right)}{\int_0^\infty e^{-rt} H_t \, dt}
\]

where \( H_t = \exp \left( \Lambda^{-1} B_2^T \Pi B_2 t \right) \)
And the link to $H_\infty$

- **Definition:** An equivalent probability measure $Q$ to $P$ is said to belong to the set $Q(\omega)$ if there exists a Radon-Nikodym derivative $q_t$ where

$$\log q_t = -\frac{1}{2} \int_0^t \theta_v^T \Sigma \theta_v \, d\nu + \int_0^t \theta_v^T \sigma \, d\xi$$

for some $\theta \in H_2$.

- Then the consumption path that solves our earlier problem solves

$$\sup_{c_t} \inf_{q_t \in Q(\omega)} \left[ \int_0^\infty e^{-rt} c_t \, dt \right] + \frac{1}{2\gamma} \left[ \int_0^\infty e^{-rt} \log q_t \, dt \right]$$

over all admissible consumption paths (i.e. budget feasible). Further the change of measure (price of risk) supporting this path is

$$\theta_t = -\gamma \sigma^T \left( \Pi B_1 + C^T RD \right) x_t$$
An example in discrete time

Timing a Value Tilt
Model: Expected Returns to a Long-Short Value Strategy

1. \( r_{t+1} \) - Returns to top 3\(^{rd}\) minus bottom 3\(^{rd}\) stocks sorted by book to price.

2. \( S_{t+1} \) - Difference in logs of book to price of top 3\(^{rd}\) minus bottom 3\(^{rd}\).

3. \( M_{t+1} \) - A weighted average of lagged returns \( r_{t+1} \)

State Equations

\[
\begin{align*}
\left( S_{t+1} - \overline{S} \right) &= \varphi_S \left( S_t - \overline{S} \right) + \eta_t \\
M_{t+1} &= r_{t+1} + \varphi_M M_t
\end{align*}
\]

Output Equation

\[
\begin{align*}
r_{t+1} &= \mu + \alpha_S \left( S_t - \overline{S} \right) + \alpha_M M_t + \varepsilon_t
\end{align*}
\]

- \( \eta < 0 \) implies lower future returns. However as \( \text{Cov}(\eta, \varepsilon) < 0 \), hedge by holding more of value portfolio.

- \( \varepsilon < 0 \) implies lower future returns due to momentum. However can hedge by holding less of value portfolio.

- Mean reversion leads to positive, momentum to negative, hedging demands
Estimate the model by maximum likelihood

1. \( r_{t+1} \) - Returns to top 3\(^{rd}\) minus bottom 3\(^{rd}\) stocks sorted by book to price.

2. \( S_{t+1} \) - Difference in logs of book to price of top 3\(^{rd}\) minus bottom 3\(^{rd}\).

3. \( M_{t+1} \) - A weighted average of lagged returns \( r_{t+1} \)

\[
\begin{align*}
\text{State Equations} \\
\begin{cases}
(S_{t+1} - 1.45) = 0.968(S_t - 1.45) + \eta_t \\
M_{t+1} = r_{t+1} + 0.75M_t 
\end{cases}
\end{align*}
\]

\[
\text{Output Equation} \\
r_{t+1} = 0.0018 + 0.028(S_t - 1.45) + 0.132(M_t - 0.008) + \varepsilon_t
\]

- Estimated on monthly data from Jan 1994 – April 2012
- The half-life of the momentum signal is just over 3 months
- There is strong negative correlation between the innovations to returns and innovations to the opportunity set.
- The \( R^2 \) of the output equation is 0.12
The Investment Objective

- The investor chooses his tilt to value. He is benchmarked to the market. Thus his fully invested portfolio returns are:

\[ r_{t+1}^{Mkt} = w_t r_{t+1} \]

We assume a constant return to the market of 4% per year, and covariance

\[
\text{Cov}
\begin{pmatrix}
\eta_t \\
\varepsilon_i \\
\varepsilon_t^{Mkt}
\end{pmatrix}
= \begin{bmatrix}
23.5\% & -0.43 & -0.15 \\
-0.43 & 11.1\% & 0.33 \\
-0.15 & 0.33 & 20\%
\end{bmatrix}
\]

Diagonal - Annualised Volatility
Off Diagonal - Correlations

- He dynamically adjust his tilts to maximise the CRA of his wealth after \( T \) periods

- Costs are calibrated so that a 1% increase in the value tilt in a month costs 0.1bps, whereas a 10% increase costs 10bps.
Target Portfolio

- Longer the horizon the lower the weight on value, due to anticipated trading costs.
- Trading momentum is more expensive, so allocation fall faster with horizon.

**Weight on Long-Short Value Portfolio**

Valuation spreads at 1 s.d (+0.26)
Momentum state at 1 s.d (+0.05)

Base when states at equilibrium

Momentum state at -1 s.d (-0.05)
Valuation spreads at -1 s.d (-0.26)

Trading Horizon
Impact of Intertemporal Hedging Motives on Target Portfolio

- Weight on value if the **Valuation Spread** is +1 s.d (0.26) above mean.
- Over a long horizon, intertemporal hedging increases weight on value.

![Graph showing correlation and weight on value over months]

- Weight on value if the **Momentum state** is +1 s.d (0.05) above mean.
- Over a long horizon, intertemporal hedging significantly reduces weight on value.

![Graph showing correlation and weight on value over months]
Impact of Costs on Target Portfolio

- Weight on value if the **Valuation Spread** is $+1 \text{ s.d} \ (0.26)$ above mean.
- Over a long horizon, costs gently reduce the weight on value.

- Weight on value if the **Momentum state** is $+1 \text{ s.d} \ (0.05)$ above mean.
- Over a long horizon, costs *significantly* reduce the weight on momentum.
Impact of Costs on Trading Speed

- Trading Speeds increase as horizon increases – as the accruing benefit increases
- Trading Speeds reduce significantly as costs increases (perhaps not very surprising!)

![Graph showing impact of costs on trading speed]

- $T = 0.01$
- $T = 0.1$
- $T = 1$

Months
Conclusions

When comparing single period optimisation to a dynamic optimisation

1. Not accounting for intertemporal hedging will
   1. Seriously overweight Momentum or trend following strategies
   2. Underweight value strategies

2. Not accounting for trading costs in relation to horizon of the strategy
   1. Overweights momentum again far more heavily than a value strategy

3. Suggests a separation strategy:
   1. Estimate the optimal long run target portfolio on the basis of costs and hedging demands
   2. Estimate the trading speed at the second stage based on maximizing net returns.
Part 3: Appendix on Trading Costs
Transactions Costs – Linear in the size of trade

- Leland (1996, 1999) and Atkinson (2004, 2010) analysed optimal rebalancing when costs are proportional to change in portfolio weights, \( TC = k/\Delta w \)

- Reasonable model if trades are small
  - Cost is of trading a block of shares is
    \[
    \frac{1}{2} \text{(Bid-Ask Spread)} \times \text{(Number of Shares)} \Rightarrow \\
    k = \frac{1}{2} \text{(Bid-Ask Spread) / (Price of Share)}
    \]

- This model of costs implies no-trade zone.
  - In zone – no trade
  - Outside – trade back to edge of the zone
Transactions Costs – No trade-zone

- Intuition is easy:
  - Costs from a non-optimal portfolio will be convex in deviation from optimality
  - Cost of trading are linear in the size of the trade

\[ \text{Loss due deviation from optimality} = TC = |\Delta w| \]

\[ \text{Gain from Trade} = \partial L \cdot \Delta w \]
\[ \text{Loss from Trade} = -k \Delta w \]

Trade if \( \partial L \cdot \Delta w > k \Delta w \)

- Capital gains can be analysed similarly (effectively just another cost)
- In multi-asset problem, numerically difficult to find borders of no trade-zone.
Transaction Costs – Large Trades

• Trading costs can be divided into:

  1. Explicit Costs: Order processing costs (compensated through the bid-ask spread)
  2. Implicit Costs: Inventory and informational costs. These cause prices to move to compensate the intermediary for risk of a large book and being the wrong side of an informed trade respectively.

• Madhaven (2000) states that

  ‘while most researchers recognize that quoted spreads are small, implicit trading costs can actually be economically significant because large trades move prices’.

• And similarly, Almgren (2005)

  ‘For large trades, the most important component of these is the impact of the trader’s own actions on the market. These costs are notoriously difficult to measure.’
The Assumption of Quadratic Transaction Costs

• A block trade $\Delta w_t$ is split into $n$ smaller trades, with each of these trades executed at prices, $p_{t+i}$ for $i=0,1..n-1$, where

\[
p_t - \tilde{p}_t = \frac{1}{2} S \text{sgn}(\Delta w_i^t)
\]

\[
\Delta \tilde{p}_{t+1} = \alpha \frac{1}{2} S \text{sgn}(\Delta w_i^t) + \varepsilon_t
\]

$\tilde{p}_t$ is mid price and $S$ is the bid-ask the spread. The coefficient $\alpha$ is the information transfer of the trade. Stoll and Huang (97) and Stoll (00) estimate $\alpha$ to be between 5-45%.

• Aggregate over the $n$ trades to find the cost of trading the block.

\[
TC = \sum_{i=0}^{i=n-1} p_{t+i} - n\tilde{p}_t = \frac{nS}{2} \left(1 - \frac{\alpha}{2}\right) + \frac{\alpha n}{2} + \text{noise}
\]

So if $n$ is large and $\alpha>0$, cost is approximately quadratic in the size of the block, $n$. 
What the research says:

- Almgren (2005) is the only work on large dataset with details of breakdown of orders to individual trades. He finds

\[ TC = k_1 (\Delta w_t)^2 + k_2 (\Delta w_t)^{1.6} \]

where \( k_1 \) and \( k_2 \) are function of stock’s volatility, liquidity, average daily volume and participation rate.

- Kissel et al (2005, 2008), I-star model:

\[ TC = k_1 (\Delta w_t)^{1.36} + k_2 (\Delta w_t)^{2.05} \]

where again \( k_1 \) and \( k_2 \) are function of stock’s volatility, liquidity and average daily volume.

**Conclusion**

Quadratic reasonable assumption for large trades - but channel is through the price impact
References


